School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet VII: Permutation Groups

1. Let G be a group and suppose that it has *equivalent* actions on the sets Ω and Ω'. Let $\rho\colon G \to \mathrm{Sym}(\Omega)$ and $\rho'\colon G \to \mathrm{Sym}(\Omega')$ be the associated permutation representations.

   (a) Show that $\ker\rho = \ker\rho'$.

   (b) Show that the permutation groups $G\rho$ and $G\rho'$ are permutation isomorphic.

2. Let $G \leqslant \mathrm{Sym}(\Omega)$ and $H \leqslant \mathrm{Sym}(\Omega')$. Suppose that G and H are permutation isomorphic via the bijection $\phi\colon \Omega \to \Omega'$ and the group isomorphism $\theta\colon G \to H$. Define

$$\omega^x = \omega^{x\theta}$$

   for each $\omega \in \Omega'$ and $x \in G$. Show that this is an action of G on Ω' and that it is equivalent to the action of G on Ω.

3. Let G be an abelian permutation group that acts transitively on the set Ω. Show that G acts regularly on Ω.

4. Let G be a group that acts upon the set Ω and let Δ be a subset of Ω. Define

$$G_{(\Delta)} = \{\, x \in G \mid \delta^x = \delta \text{ for all } \delta \in \Delta \,\},$$
$$G_{\{\Delta\}} = \{\, x \in G \mid \Delta^x = \Delta \,\}.$$

   Show that

   (a) $G_{\{\Delta\}}$ is a subgroup of G;

   (b) $G_{(\Delta)}$ is a normal subgroup of $G_{\{\Delta\}}$, and

   (c) $G_{\{\Delta\}}/G_{(\Delta)}$ is isomorphic to a subgroup of $\mathrm{Sym}(\Delta)$.

5. Let G be a group that acts upon the set Ω, let Δ be a block for G and let $\delta \in \Delta$. Show that Δ is a union of orbits of the stabilizer G_δ.

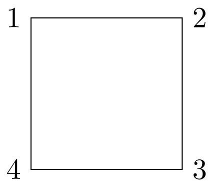
6. Let G be a group that acts transitively on a set Ω. A *G-congruence* on Ω is an equivalence relation $\sim$ such that for all $\alpha, \beta \in \Omega$ and $x \in G$

$$\alpha \sim \beta \qquad \text{if and only if} \qquad \alpha^x \sim \beta^x.$$

   (a) Suppose that Δ is a block for G and $\Sigma = \{\, \Delta^x \mid x \in G \,\}$ is the associated system of blocks. Define $\alpha \sim \beta$ if and only if α and β belong to the same set Δ^x. Show that $\sim$ is a G-congruence on Ω.

   (b) Conversely if $\sim$ is a G-congruence on Ω, show that the equivalence classes of $\sim$ are blocks for G.

7. (a) Consider the action of the dihedral group D_8 of order 8 on the set $\{1, 2, 3, 4\}$ determined by the group's action on the labelled square:



   Determine all blocks for D_8 that contain 1.

   (b) Consider the natural action of the dihedral group D_{12} of order 12 on the set $\{1, 2, 3, 4, 5, 6\}$ obtained from the corresponding clockwise labelling of the vertices of a regular hexagon. Determine all blocks for D_{12} that contain 1.

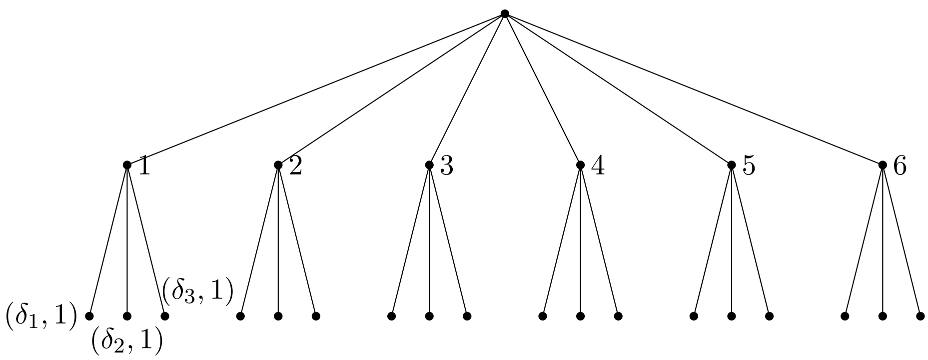
8. Let G be a group and H be a subgroup of G. Consider the permutation representation $\rho \colon G \to \mathrm{Sym}(\Omega)$ determined by the action of G on the set Ω of cosets of H.

   (a) Under what condition(s) is this action faithful?

   (b) If G is a simple group and $H \neq G$, is this action faithful?

   (c) Under what condition(s) is this action primitive?

9. Let $\Sigma = \{1, 2, \dots, n\}$ and Δ be any finite set (where we assume $n \geqslant 2$ to avoid trivial cases). Consider the following tree $\mathcal{T}$:



This tree has a root (the single vertex at the top of the diagram), one vertex attached to the root for each element in Σ (and we have labelled each vertex by the corresponding element of Σ) and, to each vertex in the middle row, one vertex in the bottom row for each element in Δ (we label the vertices in the bottom row by pairs (δ, i) for $\delta \in \Delta$ and $i \in \Sigma$). [For space reasons, in the picture only the vertices in the bottom row that are attached to the vertex labelled 1 are labelled. In the diagram, $|\Sigma| = 6$ and $|\Delta| = 3$, but you should assume n and Δ are arbitrary in this question.]

Let $W = \mathrm{Sym}(\Delta) \operatorname{wr}_\Sigma \mathrm{Sym}(\Sigma)$ be the wreath product of the symmetric group on Δ by the symmetric group on Σ and consider its imprimitive action on $\Omega = \Delta \times \Sigma$:

$$(\delta, i)^{(h_1,h_2,\dots,h_n)k} = (\delta^{h_i}, ik).$$

(a) Show that this formula defines an automorphism of the above tree $\mathcal{T}$.

(b) Show that any automorphism of $\mathcal{T}$ is determined by the application of an element of the wreath product W.

(c) Deduce that $\operatorname{Aut} \mathcal{T} \cong W$.

[**Definition:** An automorphism of a tree T is a bijection from the sets of vertices and of edges of the tree to themselves such that two vertices are joined in T if and only if their images are joined in T; that is, it is bijection from the tree to itself that preserves the structure of the graph.]

10. Suppose that $\phi\colon \Omega \to \Omega'$ is a bijection. Show that the map $\theta\colon \mathrm{Sym}(\Omega) \to \mathrm{Sym}(\Omega')$ given by
$$\theta\colon \sigma \mapsto \phi^{-1}\sigma\phi$$
is an isomorphism between these symmetric groups.

11. Let G be a group and N be a normal subgroup. Let G act on N by conjugation and let $\rho\colon G \to \operatorname{Aut} N$ be the corresponding permutation representation. Show that $\ker \rho = \mathrm{C}_G(N)$, the centralizer of N in G.

(It follows that $\mathrm{C}_G(N) \trianglelefteqslant G$ and that the quotient $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup of $\operatorname{Aut} N$.)

12. Let G be an almost simple group with minimal normal subgroup N. Show that

(a) the group $\operatorname{Inn} N$ of inner automorphisms of N is isomorphic to N;

(b) there is a subgroup $\bar{G}$ of the automorphism group of N with
$$\operatorname{Inn} N \leqslant \bar{G} \leqslant \operatorname{Aut} N$$
such that $\bar{G} \cong G$.

13. Let V be a vector space of dimension $n \geqslant 1$ over the finite field $\mathbb{F}_p$ (for some prime p) and let H be a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F}_p)$. Let $G = V \rtimes H$ viewed as a subgroup of the affine general linear group $\mathrm{AGL}_n(\mathbb{F}_p)$.

(a) Show that the induced action of G on V is transitive.

(b) Show that the stabilizer of 0 in G is $G_0 = \{\, (0, h) \mid h \in H \,\}$.

(c) Show that the action of G on V is primitive if and only if H acts irreducibly on V. [Hint: If $G_0 < K \leqslant G$, consider $W = \{\, v \in V \mid (v, 1) \in K \,\}$.]

[**Definition:** We say H acts *irreducibly* on the vector space V if there is no non-zero proper subspace W of V such that $wh \in W$ for all $w \in W$ and $h \in H$.]

14. Let $H \leqslant \mathrm{Sym}(\Delta)$ and $K \leqslant \mathrm{Sym}(\Sigma)$ where $\Sigma = \{1, 2, \ldots, n\}$, $|\Delta| > 1$ and $n \geqslant 2$. Let

$$\Omega = \Delta^{(n)} = \{ (\delta_1, \delta_2, \ldots, \delta_n) \mid \delta_1, \delta_2, \ldots, \delta_n \in \Delta \}.$$

Define an action of the wreath product $W = H \operatorname{wr}_\Sigma K$ on Ω by

$$(\delta_1, \delta_2, \ldots, \delta_n)^g = (\delta_{1k^{-1}}^{h_{1k^{-1}}}, \delta_{2k^{-1}}^{h_{2k^{-1}}}, \ldots, \delta_{nk^{-1}}^{h_{nk^{-1}}})$$

for $g = (h_1, h_2, \ldots, h_n)k$.
(So $(\delta_1, \ldots, \delta_n)^{(h_1, \ldots, h_n)} = (\delta_1^{h_1}, \ldots, \delta_n^{h_n})$ specifies the action of the base group B and then elements of K permute the entries of the elements in Ω.)

(a) Verify that this is an action of W on Ω.
(It is called the *product action* of the wreath product and, under sufficient assumptions, provides the "product type" example of a primitive permutation group appearing in the O'Nan–Scott Theorem.)

(b) Show that the product action of W on Ω is faithful.

(c) If H acts transitively on Δ, show that the product action of W on Ω is transitive.

For the remainder of the question, assume H acts transitively on Δ.

(d) If $\delta \in \Delta$, find the stabilizer $W_{\boldsymbol{\delta}}$ in W of the point $\boldsymbol{\delta} = (\delta, \delta, \ldots, \delta) \in \Omega$.

(e) If $\delta \in \Delta$ and $H_\delta < L < H$ for some subgroup L, show that $R = L \operatorname{wr}_\Sigma K$ is a subgroup of W satisfying $W_{\boldsymbol{\delta}} < R < W$.

(f) Define $D = \{ (h, h, \ldots, h) \mid h \in H \}$. If H acts regularly on Δ show that $DK = \{ dk \mid d \in D, \ k \in K \}$ is a subgroup of W satisfying $W_{\boldsymbol{\delta}} < DK < W$ for any $\delta \in \Delta$.

(g) Suppose J is an orbit of K on Σ with $J \neq \Sigma$. Define, for $\delta \in \Delta$,

$$M = \{ (h_1, h_2, \ldots, h_n) \mid h_j \in H_\delta \text{ for all } j \in J \}.$$

Show that MK is a subgroup of W satisfying $W_{\boldsymbol{\delta}} < MK < W$.

[Parts (e)–(g) show that if H acts imprimitively or regularly on Δ or if K acts intransitively on Σ, then the stabilizer of some point in Ω is *not* maximal. Hence if the product action of W on Ω is primitive, then H acts primitively but not regularly on Δ and K acts transitively. In fact, these conditions characterize (for finite permutation groups) when the product action of the wreath product is primitive.]