School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet VI: Nilpotent Groups

1. Let G be a group and, for each integer $i \geqslant 1$, let $\gamma_i(G)$ denote the ith term of the lower central series of G. Show that $\gamma_i(G)/\gamma_{i+1}(G) \leqslant \mathrm{Z}(G/\gamma_{i+1}(G))$ for all i.

2. Let G and H be nilpotent groups. Show that the direct product $G \times H$ is nilpotent.

3. Calculate the terms in the lower central series for the following groups:

   (i) S_3; (ii) D_8; (iii) D_{10}; (iv) A_4; (v) S_4.

4. Let G be a finite p-group and N be a non-trivial normal subgroup of G. Show that $\mathrm{Z}(G) \cap N \neq \mathbf{1}$.

   [Hint: Let G act by conjugation on N.]

5. Let G be a nilpotent group and N be a non-trivial normal subgroup of G. Take i to be the largest positive integer such that $\gamma_i(G) \cap N \neq \mathbf{1}$. Show that $[\gamma_i(G) \cap N, G] = \mathbf{1}$.

   Deduce that $\mathrm{Z}(G) \cap N \neq \mathbf{1}$.

6. Let G be a group and define a collection of normal subgroups $\mathrm{Z}_i(G)$ of G as follows: First set $\mathrm{Z}_0(G) = \mathbf{1}$. Then, assuming that the normal subgroup $\mathrm{Z}_i(G)$ has been defined for some integer $i \geqslant 0$, let $\mathrm{Z}_{i+1}(G)$ be the normal subgroup of G such that

   $$\mathrm{Z}_{i+1}(G)/\mathrm{Z}_i(G) = \mathrm{Z}(G/\mathrm{Z}_i(G)).$$

   (That is, $\mathrm{Z}_{i+1}(G)$ is the subgroup of G that corresponds to the centre of $G/\mathrm{Z}_i(G)$ under the Correspondence Theorem.)

   The chain

   $$\mathbf{1} = \mathrm{Z}_0(G) \leqslant \mathrm{Z}_1(G) \leqslant \mathrm{Z}_2(G) \leqslant \dots$$

   is called the *upper central series* for G.

   (a) Which subgroup of G is equal to $\mathrm{Z}_1(G)$?

   (b) Suppose that G is nilpotent and that $\gamma_{c+1}(G) = \mathbf{1}$. Show that $\mathrm{Z}_i(G) \geqslant \gamma_{c+1-i}(G)$ for $i = 0, 1, \dots, c$. Deduce that $\mathrm{Z}_c(G) = G$.

   (c) Suppose that $\mathrm{Z}_c(G) = G$ for some c. Show that $\gamma_i(G) \leqslant \mathrm{Z}_{c+1-i}(G)$ for $i = 1, 2, \dots, c+1$. Deduce that G is nilpotent.

   In conclusion, this shows that, for any group G, $\gamma_{c+1}(G) = \mathbf{1}$ if and only if $\mathrm{Z}_c(G) = G$ and hence an equivalent definition for a group to be nilpotent is that the upper central series eventually reaches G.

   [Hint for (b) and (c): Use induction on i.]

7. Calculate the terms in the upper central series for the groups listed in Question 3.

8. A *maximal subgroup* of a group G is a proper subgroup M such that there is no subgroup H with $M < H < G$.

   (a) If G is a nilpotent group, show that every maximal subgroup is normal in G.

   (b) Deduce that every maximal subgroup of a nilpotent group has index p for some prime p.

   (c) Let G be a *finite* group and suppose that P is a Sylow p-subgroup of G which is not normal in G; that is, $\mathrm{N}_G(P) < G$. Use the Frattini Argument to show that if M is a maximal subgroup of G satisfying $\mathrm{N}_G(P) \leqslant M < G$, then M is not normal in G.

   (d) Deduce that if every maximal subgroup of a finite group G is normal then G is nilpotent.

   [Thus this gives another characterization of finite nilpotent groups: A finite group is nilpotent if and only if every maximal subgroup is normal.]

9. Let G be a finite group. The Frattini subgroup $\Phi(G)$ of G is the intersection of all its maximal subgroups.

   (a) Show that $\Phi(G)$ is a characteristic subgroup of G.

   (b) What is the Frattini subgroup of a finite simple group?

   (c) What is the Frattini subgroup of an elementary abelian p-group?

   (d) If X is a subset of G such that $G = \langle X, \Phi(G) \rangle$, show that $G = \langle X \rangle$.
   [Hint: If $\langle X \rangle \neq G$, there exists a maximal subgroup that contains $\langle X \rangle$.]

   (e) Let P be a Sylow p-subgroup of the Frattini subgroup $\Phi(G)$. Use the Frattini Argument to show that P is a normal subgroup of G.

   (f) Deduce that the Frattini subgroup $\Phi(G)$ is always nilpotent.