School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet V: Soluble Groups

1. Let G and H be groups.
   (a) Show that $(G \times H)' = G' \times H'$. Deduce that $(G \times H)^{(i)} = G^{(i)} \times H^{(i)}$ for all $i \geqslant 0$.
   (b) Deduce that the direct product of two soluble groups is a soluble group. Could this result be deduced from other results proved in lectures?

2. Calculate the terms in the derived series for the following groups:

   (i) S_3; (ii) D_8; (iii) D_{10}; (iv) A_4; (v) S_5.

3. Let p be a prime number and let G be a finite p-group. Show that G is soluble.
   [Hint: Use the fact that $\mathrm{Z}(G) \neq \mathbf{1}$ for a non-trivial p-group G.]

4. Show that a group of order 1352 is soluble.
   [Hint: First find a non-trivial normal subgroup.]

5. (a) Let M and N be normal subgroups of a group G that are both soluble. Show that MN is soluble.
   [Hint: $M \trianglelefteqslant MN$. What can you say about the quotient MN/M?]
   (b) Deduce that a finite group G has a largest normal subgroup S which is soluble.
   [‘Largest’ in the usual sense of containing all others. This normal subgroup S is called the *soluble radical* of G.]
   (c) Prove that S is the unique normal subgroup of G such that S is soluble and G/S has no non-trivial abelian normal subgroup.

6. (a) Let M and N be normal subgroups of a group G such that G/M and G/N are both soluble. Show that $G/(M \cap N)$ is soluble.
   [Hint: Use the map $G \to G/M \times G/N$ given by $x \mapsto (Mx, Nx)$.]
   (b) Deduce that a finite group G has a smallest normal subgroup R such that the quotient G/R is soluble.
   [‘Smallest’ in the usual sense of being contained in all others. This normal subgroup R is called the *soluble residual* of G.]
   (c) Prove that R is the unique normal subgroup of G such that G/R is soluble and $R' = R$.
   [Hint: To show $R' = R$, remember that if H is a group with $N \trianglelefteqslant H$ such that H/N and N are soluble, then H is soluble.]

7. The following is an alternative way of proving that a minimal normal subgroup of a finite soluble group is an elementary abelian p-group.

   Let G be a finite soluble group and M be a minimal normal subgroup of G.

   (a) By considering M', prove that M is abelian. [Hint: $M' \operatorname{char} M$.]

   (b) By considering a Sylow p-subgroup of M, prove that M is a p-group for some prime p.

   (c) By considering the subgroup of M generated by all elements of order p, prove that M is an elementary abelian p-group.

8. Let $G = N \rtimes C$ be the semidirect product of a cyclic group $N = \langle x \rangle$ of order 35 by a cyclic group $C = \langle y \rangle$ of order 4 where the generator of C acts by inverting the generator of N: $y^{-1}xy = x^{-1}$.

   Find a Hall π-subgroup H of G and its normalizer $\mathrm{N}_G(H)$ when (i) $\pi = \{2, 5\}$, (ii) $\pi = \{2, 7\}$, (iii) $\pi = \{3, 5\}$ and (iv) $\pi = \{5, 7\}$.

9. Let p, q and r be distinct primes with $p < q < r$. Let G be a group of order pqr.

   (a) Show that G is soluble.

   (b) Show that G has a unique Hall $\{q, r\}$-subgroup.

   (c) How many Hall $\{p, r\}$-subgroups can G have? Can you construct examples to show that these numbers are indeed possible?

   [Hints: Review Question 2 on Problem Sheet III. You may use the fact (that we quoted but did not prove) that any two Hall π-subgroups of a finite soluble group are conjugate. Semidirect products are good ways to construct groups with specified normal subgroups.]

10. A *maximal subgroup* of G is a proper subgroup M of G such that there is no subgroup H with $M < H < G$.

    If p divides the order of the finite soluble group G, show that there is a maximal subgroup of G whose index is a power of p. [Hint: Consider Hall π-subgroups where $\pi = p'$ is the set of primes not equal to p.]

    Show, by an example, that this is false for insoluble groups.

11. Let G be a finite soluble group whose order is divisible by k distinct prime numbers. Prove there is a prime p and a Hall p'-subgroup H such that $|G| \leqslant |H|^{k/(k-1)}$.

    [Hint: Consider $|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ and consider the smallest value of $p_i^{n_i}$.]

12. Let G be a group and H and K be subgroups of G.

    (a) Show that
    $$|G : H \cap K| \leqslant |G : H| \cdot |G : K|.$$
    [Show that there is well-defined map given by $(H \cap K)x \mapsto (Hx, Kx)$ into the set of pairs of cosets of H and of K.]

    (b) If $|G : H|$ and $|G : K|$ are coprime integers, show that
    $$|G : H \cap K| = |G : H| \cdot |G : K|.$$

13. Let G be a finite soluble group and write

$$|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

where p_1, p_2, …, p_k are the distinct prime factors of the order of G. Write p_i' for the set of all primes not equal to p_i and let Q_i be a Hall p_i'-subgroup of G for $i = 1, 2, \dots, k$. [The collection $\{Q_1, Q_2, \dots, Q_k\}$ is called a *Sylow system* for G.]

(a) Show that $Q_1 \cap Q_2 \dots \cdots \cap Q_t$ is a Hall $\{p_{t+1}, \dots, p_k\}$-subgroup of G for each $t = 1$, 2, …, k. [Hint: Use induction and the previous question.]

(b) Deduce that $P_r = \bigcap_{i \neq r} Q_i$ is a Sylow p_r-subgroup of G for $r = 1, 2, \dots, k$.

(c) Show that $P_{k-1} P_k = P_k P_{k-1} = Q_1 \cap Q_2 \cap \dots \cap Q_{k-2}$.

(d) Deduce that $P_r P_s = P_s P_r$ for all $r, s \in \{1, 2, \dots, k\}$ with $r \neq s$.

[This shows that the family $\{P_1, P_2, \dots, P_k\}$ of Sylow subgroups, one for each prime divisor of $|G|$, is what is known as a *Sylow basis* for G.]