School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet IV: Semidirect Products

1. Let p be a prime number.

   (a) Show that $\operatorname{Aut} C_p \cong C_{p-1}$.

   (b) Show that $\operatorname{Aut}(C_p \times C_p) \cong \mathrm{GL}_2(\mathbb{F}_p)$ (where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denotes the field of p elements).

   [For (a), let $C_p = \langle x \rangle$. Show that an automorphism α is given by $x \mapsto x^m$ where m is a representative for a non-zero element of $\mathbb{F}_p$. Also use the fact that the multiplication group of a finite field is cyclic.

   For (b), write $C_p \times C_p$ additively and view it as a vector space over $\mathbb{F}_p$. Show that automorphisms of the group then correspond to invertible linear maps.]

2. Let $G = \langle x \rangle$ be a cyclic group. Show that $\operatorname{Aut} G$ is abelian.

3. Let N and H be groups, let $\phi \colon H \to \operatorname{Aut} N$ be a homomorphism and $G = N \rtimes_\phi H$ the semidirect product constructed using these ingredients.

   (a) Show that $h \mapsto (1, h)$ is a homomorphism $H \to G$ and that $\bar{H} = \{\, (1, h) \mid h \in H \,\}$ is a subgroup of G that is isomorphic to H.

   (b) Show that $\pi \colon (n, h) \mapsto h$ is a homomorphism $G \to H$. Determine the kernel of π. Show that $\bar{N} = \{\, (n, 1) \mid n \in N \,\}$ is a normal subgroup of G that is isomorphic to N and $G/\bar{N} \cong H$.

   (c) Show that $G = \bar{N}\bar{H}$ and $\bar{H} \cap \bar{N} = \mathbf{1}$.

   (d) Show that $(1, h)^{-1}(n, 1)(1, h) = (n^{h\phi}, 1)$ for all $h \in H$ and $n \in N$.

4. Show that the dihedral group D_{2n} is isomorphic to a semidirect product of a cyclic group of order n by a cyclic group of order 2. What is the associated homomorphism $\phi \colon C_2 \to \operatorname{Aut} C_n$?

5. Show that the quaternion group Q_8 may not be decomposed (in a non-trivial way) as a semidirect product.

   [Hint: How many elements of order 2 does Q_8 contain?]

6. Show that the symmetric group S_4 of degree 4 is isomorphic to a semidirect product of the Klein 4-group V_4 by the symmetric group S_3 of degree 3.

   Show that S_4 is also isomorphic to a semidirect product of the alternating group A_4 by a cyclic group of order 2.

7. Let $G = F \rtimes T$ be the semidirect product of a cyclic group $F = \langle x \rangle$ of order 5 by $T = \langle y, z \rangle \cong C_2 \times C_2$ where $y^{-1}xy = x$ and $z^{-1}xz = x^4$.

   Show that G is isomorphic to the dihedral group D_{20} of order 20.

   [Hint: What is the order of the element xy?]

8. Let G be a group of order pq where p and q are primes with $p < q$.

   (a) If p does not divide $q - 1$, show that $G \cong C_{pq}$, the cyclic group of order pq.

   (b) If p does divide $q - 1$, show that there are essentially two different groups of order pq.

9. Classify the groups of order 52 up to isomorphism.

10. Show that a group of order 30 is isomorphic to one of

$$C_{30}, \quad C_3 \times D_{10}, \quad D_6 \times C_5, \quad D_{30}.$$

11. Classify the groups of order 98 up to isomorphism.

12. Classify the groups of order 117 up to isomorphism.

13. Let H and N be groups and let $\phi, \psi \colon H \to \operatorname{Aut} N$ be two homomorphisms. Suppose that there exists $\alpha \in \operatorname{Aut} N$ and $\beta \in \operatorname{Aut} H$ such that $(h^\beta)\phi = \alpha^{-1}(h\psi)\alpha$ for all $h \in H$. Show that $(n, h) \mapsto (n^\alpha, h^\beta)$ is an isomorphism from $N \rtimes_\phi H$ to $N \rtimes_\psi H$.

    [In this question, we are writing x^γ for the image of an element x of a group under an automorphism γ of the group. This is consistent with our notation in the semidirect product where we write $n^{h\phi}$ for the image of $n \in N$ under the automorphism $h\phi$.]

14. Let G and H be groups where G acts on the set $\Omega = \{1, 2, \dots, n\}$. Consider the wreath product $W = H \operatorname{wr}_\Omega G$ determined by this action and let B be the base group of W.

    We shall view G as a subgroup of W by identifying it with the usual choice of complement to B naturally occurring in this semidirect product.

    (a) If N is a normal subgroup of H, show that the direct product $N \times N \times \cdots \times N$ of n copies of N is a normal subgroup of W.

    (b) Show that an element $(h_1, h_2, \dots, h_n)$ of B commutes with all elements of G if and only if $h_i = h_j$ whenever i and j lie in the same orbit of G on Ω.

    (c) Suppose that $H \neq \mathbf{1}$ and that G acts transitively and also *faithfully* on Ω; that is, the kernel of the associated permutation representation is trivial. Show that the centre of W is

$$\mathrm{Z}(W) = \{\, (h, h, \dots, h) \mid h \in \mathrm{Z}(H) \,\}.$$

    (d) Suppose that $H \neq \mathbf{1}$ and that G acts faithfully on Ω. If N is a non-trivial normal subgroup of W, show that $N \cap B \neq \mathbf{1}$.

    [Hint: If $N \cap B = \mathbf{1}$, what does this say about commutators $[x, b]$ where $x \in N$ and $b \in B$?]