School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet III: Composition Series, Chief Series, the Jordan–Hölder Theorem, Characteristic Subgroups, and Minimal Normal Subgroups

1. Let G be a group and N be a normal subgroup of G. If G/N and N both have composition series, show that G has a composition series and that the set of composition factors of G equals the union of those of G/N and those of N.

   [Hint: Use the Correspondence Theorem to lift the terms of a composition series of G/N to a chain of subgroups between G and N.]

2. Let p, q and r be prime numbers with $p < q < r$. Let G be a group of order pqr.

   (a) Suppose that G does not have a unique Sylow r-subgroup. Show that it has a unique Sylow q-subgroup Q.

   [Hint: How many elements of order r must there be and how many elements of order q if there were no normal Sylow q-subgroup?]

   (b) Show that G/Q has a unique Sylow r-subgroup of the form K/Q where $K \trianglelefteq G$ and $|K| = qr$.

   (c) Show that K has a unique Sylow r-subgroup and deduce that, in fact, G has a unique Sylow r-subgroup R (contrary to the assumption in (a)).

   [Hint: Why is a Sylow r-subgroup of K also a Sylow r-subgroup of G? Remember that all the Sylow r-subgroups of G are conjugate.]

   Let R be the unique Sylow r-subgroup of G. [We have established in parts (a)–(c) that the group G has a unique Sylow r-subgroup.]

   (d) Show that G/R has a unique Sylow q-subgroup.

   (e) Deduce that G has a composition series

   $$G = G_0 > G_1 > G_2 > G_3 = \mathbf{1}$$

   where $|G_1| = qr$ and $|G_2| = r$. Up to isomorphism, what are the composition factors of G?

3. Let p be a prime number and let G be a non-trivial p-group. Show that G has a chain of subgroups

   $$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

   such that G_i is a normal subgroup of G and $|G : G_i| = p^i$ for $i = 0, 1, \ldots, n$.

   What are the composition factors of G?

   [Hint: Use the fact that $\mathrm{Z}(G) \neq 1$ to produce an element $x \in \mathrm{Z}(G)$ of order p and then consider the quotient group G/K where $K = \langle x \rangle$.]

4. The dihedral group D_8 has seven different composition series. Find all seven.

5. How many different composition series does the quaternion group Q_8 have?

6. Let $n \geqslant 5$. Show that the alternating group A_n is the unique non-trivial proper normal subgroup of the symmetric group S_n.

   Deduce that S_n has precisely one composition series.

   [Hint: If $N \trianglelefteqslant S_n$ with $\mathbf{1} < N < S_n$, consider $N \cap A_n$ and $N \cdot A_n$.]

7. Let G be a group and let $\mathcal{S} = \mathcal{S}(G)$ be a collection of subgroups of G satisfying

   ($\mathcal{S}1$) $\mathbf{1}, G \in \mathcal{S}$;

   ($\mathcal{S}2$) if $H, K \in \mathcal{S}$ then $H \cap K \in \mathcal{S}$;

   ($\mathcal{S}3$) if $H, K, L \in \mathcal{S}$ with $H \leqslant L$ and $K \trianglelefteqslant L$, then $HK \in \mathcal{S}$.

   Let N be a normal subgroup of G that belongs to $\mathcal{S}$ and suppose that

   $$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

   is a series for G with terms in $\mathcal{S}$ that is maximal. Define $N_i = N \cap G_i$ for $i = 0, 1, \ldots, n$.

   (a) Show that N_{i+1} is a normal subgroup of N_i for $i = 0, 1, \ldots, n-1$.

   (b) Show that $N_i G_{i+1} \in \mathcal{S}$ and hence that $N_i G_{i+1}$ is either equal to G_i or G_{i+1}. Show, moreover, that if $N_i G_{i+1} = G_{i+1}$, then $N_i = N_{i+1}$.

   (c) Use the Second Isomorphism Theorem to show that

   $$N_i / N_{i+1} \cong N_i G_{i+1} / G_{i+1}.$$

   [Hint: Note that $N_{i+1} = G_{i+1} \cap (G_i \cap N)$.]

   (d) Show that a member R of $\mathcal{S}$ satisfying $N_{i+1} \leqslant R \trianglelefteqslant N_i$, either equals N_i or N_{i+1}. [Hint: Consider the image of R/N_{i+1} under the isomorphism given in part (c).]

   (e) By deleting repeats in the series (N_i), deduce that N possesses a series with terms in $\mathcal{S}$ that is maximal.

8. Let G be a group, N be a normal subgroup of G and suppose that

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

is a composition series for G. Define $Q_i = G_i N/N$ for $i = 0, 1, \ldots, n$.

   (a) Show that Q_i is a subgroup of G/N such that Q_{i+1} is a normal subgroup of Q_i for $i = 0, 1, \ldots, n-1$.

   (b) Show that
   $$Q_i/Q_{i+1} \cong \frac{G_i/G_{i+1}}{(G_i \cap N)G_{i+1}/G_{i+1}}.$$
   [Hint: Use the Third Isomorphism Theorem, note $G_i N = G_i(G_{i+1}N)$ because $G_{i+1} \leqslant G_i$, use the Second Isomorphism Theorem and then apply Dedekind's Modular Law to deduce information about $G_{i+1}N \cap G_i$.]

   (c) Show that $(G_i \cap N)G_{i+1}$ is a normal subgroup of G_i containing G_{i+1}.
   Deduce that $(G_i \cap N)G_{i+1}/G_{i+1}$ is either equal to G_i/G_{i+1} or to the trivial group.
   Hence show that the quotient on the right hand side in (b) is either trivial or isomorphic to G_i/G_{i+1}.

   (d) Deduce that G/N possesses a composition series.

9. Show that $\mathbf{1}$ and G are always characteristic subgroups of a group G.

   Give an example of a group G with a normal subgroup N such that N is not a characteristic subgroup of G.

10. Let G be a finite group and suppose that P is a Sylow p-subgroup of G (for some prime p) which is actually normal in G. Prove that P is a characteristic subgroup of G.

    [Hint: Where does an automorphism map a Sylow p-subgroup?]

11. Let G be any group and let n be an integer. Define

$$G^n = \langle\, g^n \mid g \in G \,\rangle,$$

the subgroup of G generated by all its nth powers. Show that G^n is a characteristic subgroup of G. Show that every element in the quotient group G/G^n has order dividing n.

12. (a) Give an example of a group G with subgroups K and H such that $K \trianglelefteqslant H$, $H \trianglelefteqslant G$ but $K \ntrianglelefteq G$.

   (b) Give an example of a group G with a characteristic subgroup H and a homomorphism $\phi \colon G \to K$ such that $H\phi$ is not a characteristic subgroup of $G\phi$.

   (c) Give an example of a group G with subgroups H and L such that $H \leqslant L \leqslant G$, $H \operatorname{char} G$, but H is not a characteristic subgroup of L.

   (d) Give an example of a group G with subgroups K and H such that $K \trianglelefteqslant H$, H is a characteristic subgroup of G but $K \ntrianglelefteq G$.

   [Hint: A little thought should tell you that $|G| \geqslant 8$ is required for many of these examples. There are examples without making the groups considerably larger than this minimum.]

13. Let S_1, S_2, …, S_n be non-abelian simple groups and let

$$G = S_1 \times S_2 \times \cdots \times S_n.$$

We shall identify each simple group S_i with the copy

$$\bar{S}_i = \{ (1, \dots, 1, x, 1, \dots, 1) \mid x \in S_i \}$$

that occurs as a normal subgroup of the direct product.

Prove that a non-trivial normal subgroup of G necessarily contains one of the direct factors S_i.

[Hint: If N is a non-trivial normal subgroup of G, choose some non-identity element $(x_1, x_2, \dots, x_n) \in N$. Consider conjugating this element by an appropriate choice of element of the form $(1, \dots, 1, g, 1, \dots, 1)$.]

Hence show that every normal subgroup of G has the form

$$S_{i_1} \times S_{i_2} \times \cdots \times S_{i_k}$$

for some subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$.

Now suppose S_1, S_2, …, S_n are *abelian* simple groups. Is it still true that every normal subgroup of the direct product has this form?

14. Let S be a simple group and $G = S \times S \times \cdots \times S$ be the direct product of d copies of S. Show that G is characteristically simple.

[This will establish the converse of Theorem 3.22. Consider two cases: First when S is a cyclic group of prime order p, in which case it might help to view G as a vector space of dimension d over the field $\mathbb{F}_p$ of p elements. Second when S is a non-abelian simple group, in which case Question 13 will be useful.]