MRQ 2022

School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet II: Group Actions

1. (a) How many different ways can the cyclic group C_3 of order three act on the set $\{1, 2, 3, 4\}$?

   (b) How many different ways can the cyclic group C_4 of order four act on the set $\{1, 2, 3\}$?

   [Consider orbit decompositions and apply the Orbit-Stabilizer Theorem.]

2. Let G be a group and Ω be a set such that G acts on Ω. If $\Gamma = \omega^G$ is the orbit of a point $\omega \in \Omega$ under the action of G, show that there is an induced action of G on Γ and that this action is transitive.

3. (a) Let i_1, i_2, $\dots$, i_k be distinct points in $\Omega = \{1, 2, \dots, n\}$ and let σ be a permutation in S_n. By considering the effect on various points in Ω, or otherwise, show that

   $$\sigma^{-1}(i_1\ i_2\ \dots\ i_k)\sigma = (i_1\sigma\ i_2\sigma\ \dots\ i_k\sigma).$$

   [To reduce the number sub/superscripts, in this formula we are writing $i\sigma$ for the image of a point i under the permutation σ.]

   Deduce that two permutations in S_n are conjugate if and only if they have the same cycle structure.

   (b) Give a list of representatives for the conjugacy classes in S_5. How many elements are there in each conjugacy class? Hence calculate the order of and generators for the centralizers of these representatives.

4. Let G be a group and let Γ and Δ be sets such that G acts on Γ and on Δ. Define

   $$(\gamma, \delta)^x = (\gamma^x, \delta^x)$$

   for all $\gamma \in \Gamma$, $\delta \in \Delta$ and $x \in G$. Verify that this is an action of G on the set $\Gamma \times \Delta$.

   Verify that the stabilizer of the pair (γ, δ) in this action equals the intersection of the stabilizers G_γ and G_δ (these being the stabilizers under the actions of G on Γ and Δ, respectively).

   If G acts transitively on the non-empty set Ω, show that

   $$\{\, (\omega, \omega) \mid \omega \in \Omega \,\}$$

   is an orbit of G on $\Omega \times \Omega$. Deduce that G acts transitively on $\Omega \times \Omega$ if and only if $|\Omega| = 1$.

5. (a) There is a natural action of S_n on $\Omega = \{1, 2, \dots, n\}$. Show this action is transitive. How many orbits does S_n have on $\Omega \times \Omega$?

   (b) Repeat part (a) with the action of the alternating group A_n on Ω.

6. The symmetric group S_n of degree n acts in its usual definition on the set $\Omega = \{1, 2, \ldots, n\}$. In this question, we write $i\sigma$ for the image of $i \in \Omega$ under a permutation σ to reduce the need for double sub/superscripts.

   We now define an action of S_n on another set.

   Let X be any set and define

   $$\Delta = X^{(n)} = \{\, (x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in X \,\}.$$

   If σ is a permutation in the symmetric group S_n and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Delta$, define

   $$\boldsymbol{x}^\sigma = (x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, \ldots, x_{n\sigma^{-1}}).$$

   [Consequently, if a permutation σ moves i to j (that is, $i\sigma = j$) then applying σ to a sequence $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ moves the content of position i in $\boldsymbol{x}$ into position j in $\boldsymbol{x}^\sigma$. This is why the inverse σ^{-1} appears in the formula.]

   Show that this defines an action of S_n on Δ.

7. Let G be a group and H be a subgroup of G. Show that the normalizer $\mathrm{N}_G(H)$ of H is the largest subgroup of G in which H is a normal subgroup.

   [By *largest*, we mean that if L is any subgroup of G in which H is normal, then $L \leqslant \mathrm{N}_G(H)$. So you should check that (i) $H \trianglelefteqslant \mathrm{N}_G(H)$ and (ii) if $H \trianglelefteqslant L$ then $L \leqslant \mathrm{N}_G(H)$.]

8. (a) Let G be a group and consider the right regular action of G on itself. (Recall this is given by $(g, x) \mapsto gx$.)
   Show that G acts transitively on itself by this formula. If $g \in G$, determine the stabilizer of g under this action.

   (b) Let G be a group and H be a subgroup of G. Let H act on G via the induced action obtained from the right regular action; that is,

   $$\begin{aligned} G \times H &\to G \\ (g, x) &\mapsto gx. \end{aligned}$$

   Show this is an action of H on G. Determine the orbits under this action of H. What theorem from group theory can you deduce from this action?

   (c) Let G be any group. Define $*$ by the formula

   $$g * x = x^{-1} g$$

   and view this as applying x to the element g. Show that this defines an action of G on itself. (This is the *left regular action* of G on itself.)
   Is this action transitive? If $g \in G$, determine the stabilizer of g under this action.
   Let H be a subgroup of G and consider the induced action of H on G obtained from the left regular action:

   $$\begin{aligned} G \times H &\to G \\ (g, x) &\mapsto g * x = x^{-1} g. \end{aligned}$$

   Determine the orbits under this action of H. What theorem from group theory can you deduce?

9. If H is a subgroup of G of index n, show that the index of the core of H in G divides $n!$.

   [Hint: Use the action on cosets.]

10. Let G be a group and let G act on itself by conjugation. Show that the kernel of the associated permutation representation $\rho\colon G \to \mathrm{Sym}(G)$ is the centre $\mathrm{Z}(G)$ of G. Deduce that $\mathrm{Z}(G)$ is a normal subgroup of G.

11. Let G be a group and let $\mathrm{Aut}\,G$ denote the set of all automorphisms of G. Show that $\mathrm{Aut}\,G$ forms a group under composition.

    For $g \in G$, let $\tau_g\colon G \to G$ be the map given by conjugation by g; that is,

    $$\tau_g\colon x \mapsto g^{-1}xg \qquad \text{for all } x \in G.$$

    Show that the map $\tau\colon g \mapsto \tau_g$ is a homomorphism $\tau\colon G \to \mathrm{Aut}\,G$. What is the kernel of τ?

    Write $\mathrm{Inn}\,G$ for the image of τ. Thus $\mathrm{Inn}\,G$ is the set of *inner automorphisms* of G. Show that $\mathrm{Inn}\,G$ is a normal subgroup of $\mathrm{Aut}\,G$. [Hint: Calculate the effect of $\phi^{-1}\tau_g\phi$ on an element x, where $\phi \in \mathrm{Aut}\,G$, $\tau_g \in \mathrm{Inn}\,G$ and $x \in G$.]

12. Show that there is no simple group of order equal to each of the following numbers:

(i) 30;	(ii) 48;	(iii) 54;	(iv) 66;	(v) 72;
(vi) 84;	(vii) 104;	(viii) 132;	(ix) 150;	(x) 392.

    [Note: Some of these can be solved using Sylow's Theorem and methods from *MT4003* alone, while others require the technology of group actions. They are not necessarily in increasing order of difficulty!]

13. Let G be a simple group of order 60.

    (a) Show that G has no proper subgroup of index less than 5. Show that if G has a subgroup of index 5, then $G \cong A_5$.

    [Hint: If H is a subgroup of index k, let G act on the set of cosets to produce a permutation representation. What do we know about the kernel?]

    (b) Let S and T be distinct Sylow 2-subgroups of G. Show that if $x \in S \cap T$ then $|\mathrm{C}_G(x)| \geqslant 12$. Deduce that $S \cap T = \mathbf{1}$. [Hint: What do you know about S and T from their orders?]

    (c) Deduce that G has at most five Sylow 2-subgroups and hence that indeed $G \cong A_5$.

    Thus we have shown that there is a unique simple group of order 60 up to isomorphism.

14. Let p be a prime number, P be a finite p-group and let P act on a finite set Ω. Define

    $$\mathrm{Fix}_P(\Omega) = \{\, \omega \in \Omega \mid \omega^x = \omega \text{ for all } x \in P \,\},$$

    the set of fixed-points in the action. By considering the orbits for the action, show that

    $$|\mathrm{Fix}_P(\Omega)| \equiv |\Omega| \pmod{p}.$$

15. Let G be a finite group, let p be a prime number and write $|G| = p^n m$ where p does not divide m. The purpose of this question is to use group actions to show G has a subgroup of order p^n; that is, G has a Sylow p-subgroup. [Do *not* use Sylow's Theorem! The goal here is to establish one part of that theorem.]

   (a) Let Ω be the collection of all *subsets* of G of size p^n:

   $$\Omega = \{\, S \subseteq G \mid |S| = p^n \,\}.$$

   Show that

   $$\begin{aligned} |\Omega| &= \binom{p^n m}{p^n} \\ &= m \left(\frac{p^n m - 1}{1} \right) \left(\frac{p^n m - 2}{2} \right) \cdots \left(\frac{p^n m - p^n + 2}{p^n - 2} \right) \left(\frac{p^n m - p^n + 1}{p^n - 1} \right). \end{aligned}$$

   (b) Let j be an integer with $1 \leqslant j \leqslant p^n - 1$. Show that if the prime power p^i divides j, then p^i divides $p^n m - j$. Conversely show that if p^i divides $p^n m - j$, then p^i divides j. Deduce that $|\Omega|$ is not divisible by p.

   (c) Show that we may define a group action of G on Ω by

   $$\begin{aligned} \Omega \times G &\to \Omega \\ (S, x) &\mapsto Sx = \{\, ax \mid a \in S \,\}. \end{aligned}$$

   (d) Express Ω as a disjoint union of orbits:

   $$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k.$$

   Show that p does not divide $|\Omega_i|$ for some i.

   (e) Let $S \in \Omega_i$ and let $P = G_S$, the stabilizer of S in the action of G on Ω. Show that p does not divide $|G : P|$ and deduce that p^n divides $|P|$.

   (f) Fix $a_0 \in S$. Explain why $a_0 x \in S$ for all $x \in P$. Show that $x \mapsto a_0 x$ is an injective map $P \to S$. Deduce that $|P| \leqslant p^n$.

   (g) Conclude that P is a Sylow p-subgroup of G.

16. Let G be a finite group, p be a prime number and P be a Sylow p-subgroup of G (which we may assume exists by the previous question). The purpose of this question is to use group actions to establish the majority of the remaining parts of Sylow's Theorem (e.g., the information about conjugacy of Sylow subgroups). [Again do *not* use Sylow's Theorem!]

   Let $\Sigma = \{ P^g \mid g \in G \}$ be the set of conjugates of P in G. Note that this set consists of *some* of the Sylow p-subgroups of G.

   (a) If $Q \in \Sigma$ and H is a p-subgroup of G, show that $H \leqslant \mathrm{N}_G(Q)$ if and only if $H \leqslant Q$. [Hint: If $H \leqslant \mathrm{N}_G(Q)$, consider the product HQ. Why is it a subgroup? What is its order?]

   (b) If H is any p-subgroup of G, define an action of H on Σ by:

   $$\begin{aligned} \Sigma \times H &\to \Sigma \\ (Q, x) &\mapsto Q^x \end{aligned}$$

   Show that $Q \in \mathrm{Fix}_H(\Sigma)$ if and only if $H \leqslant Q$.

   (c) By taking $H = P$ in the previous part, show that $|\Sigma| \equiv 1 \pmod{p}$. [Hint: Use Question 14]

   (d) Let R be any Sylow p-subgroup of G. By taking $H = R$ in part (b), show that $R \in \Sigma$. Deduce that all Sylow p-subgroups of G are conjugate in G.

   (e) Use part (b) once more to deduce that every p-subgroup of G is contained in a Sylow p-subgroup.