School of Mathematics and Statistics

MT5864 Advanced Group Theory

Problem Sheet I: Review/Revision

1. Let G be a group and H, K and L be subgroups of G with $K \leqslant L$. Show that
$$HK \cap L = (H \cap L)K.$$
[This result is known as **Dedekind's Modular Law**.]

2. Let H and K be subgroups of a group G. Define
$$HK = \{\, hk \mid h \in H,\ k \in K \,\}.$$
   (a) Show that HK is a subgroup of G if and only if $HK = KH$.
   (b) Show that if K is a normal subgroup of G, then HK is a subgroup of G.
   (c) Give an example of a group G and two subgroups H and K such that HK is not a subgroup of G.
   (d) Give an example of a group G and two subgroups H and K such that HK is a subgroup of G but neither H nor K are normal subgroups of G. [Ideally, find examples with $H \nleqslant K$ and $K \nleqslant H$.]

3. Let M and N be normal subgroups of G. Show that $M \cap N$ and MN are normal subgroups of G.

4. Let G be a group and H be a subgroup of G.
   (a) If x and y are elements of G, show that $Hx = Hy$ if and only if $x \in Hy$.
   (b) Suppose T is a subset of G containing precisely one element from each (right) coset of H in G (such a set T is called a (*right*) *transversal* to H in G and has the property that $|T| = |G : H|$). Deduce that $\{\, Ht \mid t \in T \,\}$ is the set of all (right) cosets of H in G with distinct elements of T defining distinct cosets.

5. Let G be a (not necessarily finite) group with two subgroups H and K such that $K \leqslant H \leqslant G$. The purpose of this question is to establish the index formula

$$|G:K| = |G:H| \cdot |H:K|.$$

Let T be a transversal to K in H and U be a transversal to H in G.

(a) By considering the coset Hg or otherwise, show that if g is an element of G, then $Kg = Ktu$ for some $t \in T$ and some $u \in U$.

(b) If $t, t' \in T$ and $u, u' \in U$ with $Ktu = Kt'u'$, show that $t = t'$ and $u = u'$. [Hint: First show $Hu = Hu'$.]

(c) Deduce that $TU = \{ tu \mid t \in T,\ u \in U \}$ is a transversal to K in G and that

$$|G:K| = |G:H| \cdot |H:K|.$$

6. Let G be a group and H be a subgroup of G.

(a) Show that H is a normal subgroup of G if and only if $Hx = xH$ for all $x \in G$.

(b) Show that if $|G:H| = 2$, then H is a normal subgroup of G.

7. Let H be a subgroup of the symmetric group S_n of index 2. Show that $H = A_n$. [Hint: Show that H contains all squares of elements in S_n.]

8. Give an example of a finite group G and a divisor m of $|G|$ such that G has no subgroup of order m.

9. Let $G = \langle x \rangle$ be a cyclic group.

(a) If H is a non-identity subgroup of G, show that H contains an element of the form x^k with $k > 0$.

Choose k to be the smallest positive integer such that $x^k \in H$. Show that every element in H has the form x^{kq} for some $q \in \mathbb{Z}$ and hence that $H = \langle x^k \rangle$. [Hint: Use the Division Algorithm.]

Deduce that every subgroup of a cyclic group is also cyclic.

(b) Suppose now that G is cyclic of order n. Let H be the subgroup considered in part (a), so that $H = \langle x^k \rangle$ where k is the smallest positive integer such that $x^k \in H$, and suppose that $|H| = m$.

Show that k divides n. [Hint: Why does $x^n \in H$?]

Show that $o(x^k) = n/k$ and deduce that $m = n/k$.

Conclude that, if G is a cyclic group of finite order n, then G has a unique subgroup of order m for each positive divisor m of n.

(c) Suppose now that G is cyclic of infinite order. Let H be the subgroup considered in part (a), so that $H = \langle x^k \rangle$ where k is the smallest positive integer such that $x^k \in H$. Show that $\{1, x, x^2, \dots, x^{k-1}\}$ is a transversal to H in G. Deduce that $|G:H| = k$. [Hint: Use the Division Algorithm again.]

Conclude that, if G is a cyclic group of infinite order, then G has a unique subgroup of index k for each positive integer k and that every non-trivial subgroup of G is equal to one of these subgroups.

10. The *dihedral group* D_{2n} of order $2n$ is generated by the two permutations

$$\alpha = (1\ 2\ 3\ \dots\ n), \qquad \beta = (2\ n)(3\ n{-}1)\cdots.$$

(a) Show that α generates a normal subgroup of D_{2n} of index 2.

(b) Show that every element of D_{2n} can be expressed in the form $\alpha^i \beta^j$ where i and j are integers with $0 \leqslant i \leqslant n-1$ and $j \in \{0, 1\}$.

(c) Show that every element in D_{2n} which does not lie in $\langle \alpha \rangle$ has order 2.

11. The *quaternion group* Q_8 of order 8 consists of eight elements

$$1, -1, i, -i, j, -j, k, -k$$

with multiplication given by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

(a) Show that Q_8 is generated by i and j.

(b) Show that $\langle i \rangle$ is a normal subgroup of Q_8 of index 2.

(c) Show that every element of Q_8 can be written as $i^m j^n$ where $m \in \{0, 1, 2, 3\}$ and $n \in \{0, 1\}$.

(d) Show that every element in Q_8 which does not lie in $\langle i \rangle$ has order 4.

(e) Show that Q_8 has a unique element of order 2.

12. Recall (from *MT4003*) that if $G_1, G_2, \dots, G_n$ are a collection of groups, then their direct product is the set

$$G_1 \times G_2 \times \cdots \times G_n = \{\, (x_1, x_2, \dots, x_n) \mid x_i \in G_i \text{ for each } i \,\}$$

with componentwise multiplication:

$$(x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n) = (x_1 y_1, x_2 y_2, \dots, x_n y_n).$$

You may assume that this does indeed define a group.

(a) Let G be a group possessing normal subgroups $H_1, H_2, \dots, H_n$ satisfying

(1) $G = H_1 H_2 \dots H_n$ (that is, every element of G can be expressed as $x_1 x_2 \dots x_n$ where $x_i \in H_i$ for each i) and

(2) $H_i \cap H_1 \dots H_{i-1} H_{i+1} \dots H_n = \mathbf{1}$ for each i.

Show that the map $(x_1, x_2, \dots, x_n) \mapsto x_1 x_2 \dots x_n$ is an isomorphism from the direct product $H_1 \times H_2 \times \cdots \times H_n$ to G.

[Hint: In preparation for showing this is a homomorphism, it will probably help to consider the element $x^{-1} y^{-1} x y$ where $x \in H_i$ and $y \in H_j$ with $i \neq j$.]

(b) Give an example of a group G with three normal subgroups H_1, H_2 and H_3 such that $G = H_1 H_2 H_3$ and $H_i \cap H_j = \mathbf{1}$ for $i \neq j$ but where $G \not\cong H_1 \times H_2 \times H_3$.

In your example, why is the map $(x_1, x_2, x_3) \mapsto x_1 x_2 x_3$ *not* an isomorphism?

13. Let G be a finite group, N be a normal subgroup of G and P be a Sylow p-subgroup of G.

   (a) Show that $P \cap N$ is a Sylow p-subgroup of N.

   (b) Show that PN/N is a Sylow p-subgroup of G/N.

   [Hint: Show that the subgroup is of order a power of p and has index not divisible by p. In both parts expect to use the formula for the order of PN and the fact that P already has the required property as a subgroup of G.]

14. Let G be a finite group, p be a prime number dividing the order of G, and let P be a Sylow p-subgroup of G. Define

$$O_p(G) = \bigcap_{g \in G} P^g.$$

   Show that $O_p(G)$ is the largest normal p-subgroup of G.