School of Mathematics and Statistics

MT5824 Topics in Groups

Problem Sheet I: Revision and Re-Activation

1. Let H and K be subgroups of a group G. Define

$$HK = \{\, hk \mid h \in H,\ k \in K \,\}.$$

(a) Show that HK is a subgroup of G if and only if $HK = KH$.

(b) Show that if K is a normal subgroup of G, then HK is a subgroup of G.

(c) Give an example of a group G and two subgroups H and K such that HK is not a subgroup of G.

(d) Give an example of a group G and two subgroups H and K such that HK is a subgroup of G but neither H nor K are normal subgroups of G.

2. Let M and N be normal subgroups of G. Show that $M \cap N$ and MN are normal subgroups of G.

3. Let G be a group and H be a subgroup of G.

(a) If x and y are elements of G, show that $Hx = Hy$ if and only if $x \in Hy$.

(b) Suppose T is a subset of G containing precisely one element from each (right) coset of H in G (such a set T is called a (*right*) *transversal* to H in G and has the property that $|T| = |G : H|$). Deduce that $\{\, Ht \mid t \in T \,\}$ is the set of all (right) cosets of H in G with distinct elements of T defining distinct cosets.

4. Let G be a (not necessarily finite) group with two subgroups H and K such that $K \leqslant H \leqslant G$. The purpose of this question is to establish the index formula

$$|G : K| = |G : H| \cdot |H : K|.$$

Let T be a transversal to K in H and U be a transversal to H in G.

(a) By considering the coset Hg or otherwise, show that if g is an element of G, then $Kg = Ktu$ for some $t \in T$ and some $u \in U$.

(b) If $t, t' \in T$ and $u, u' \in U$ with $Ktu = Kt'u'$, first show that $Hu = Hu'$ and deduce $u = u'$, and then show that $t = t'$.

(c) Deduce that $TU = \{\, tu \mid t \in T,\ u \in U \,\}$ is a transversal to K in G and that

$$|G : K| = |G : H| \cdot |H : K|.$$

(d) Show that this formula follows immediately from Lagrange's Theorem if G is a finite group.

5. Let G be a group and H be a subgroup of G.

(a) Show that H is a normal subgroup of G if and only if $Hx = xH$ for all $x \in G$.

(b) Show that if $|G : H| = 2$, then H is a normal subgroup of G.

6. Give an example of a finite group G and a divisor m of $|G|$ such that G has no subgroup of order m.

7. Let $G = \langle x \rangle$ be a cyclic group.

(a) If H is a non-identity subgroup of G, show that H contains an element of the form x^k with $k > 0$.
Choose k to be the smallest positive integer such that $x^k \in H$. Show that every element in H has the form x^{kq} for some $q \in \mathbb{Z}$ and hence that $H = \langle x^k \rangle$. [Hint: Use the Division Algorithm.]
Deduce that every subgroup of a cyclic group is also cyclic.

(b) Suppose now that G is cyclic of order n. Let H be the subgroup considered in part (a), so that $H = \langle x^k \rangle$ where k is the smallest positive integer such that $x^k \in H$, and suppose that $|H| = m$.
Show that k divides n. [Hint: Why does $x^n \in H$?]
Show that $o(x^k) = n/k$ and deduce that $m = n/k$.
Conclude that, if G is a cyclic group of finite order n, then G has a unique subgroup of order m for each positive divisor m of n.

(c) Suppose now that G is cyclic of infinite order. Let H be the subgroup considered in part (a), so that $H = \langle x^k \rangle$ where k is the smallest positive integer such that $x^k \in H$.
Show that $\{1, x, x^2, \dots, x^{k-1}\}$ is a transversal to H in G. Deduce that $|G : H| = k$. [Hint: Use the Division Algorithm to show that if $n \in \mathbb{Z}$, then $x^n \in Hx^r$ where $0 \leqslant r < k$.]
Conclude that, if G is a cyclic group of infinite order, then G has a unique subgroup of index k for each positive integer k and that every non-trivial subgroup of G is equal to one of these subgroups.

8. Let V_4 denote the *Klein 4-group*: that is $V_4 = \{1, a, b, c\}$ where $a = (1\,2)(3\,4)$, $b = (1\,3)(2\,4)$ and $c = (1\,4)(2\,3)$ (permutations of four points). Find three distinct subgroups H_1, H_2 and H_3 of V_4 of order 2. Show that $H_i \cap H_j = \mathbf{1}$ for all $i \neq j$ and $V_4 = H_i H_j$ for all i and j.

[Note that I am using the more conventional notation V_4 for the Klein 4-group, rather than the less frequently used K_4 from MT4003. Here V stands for *Vier-gruppe*.]

9. The *dihedral group* D_{2n} of order $2n$ is generated by the two permutations

$$\alpha = (1\,2\,3\,\ldots\,n), \qquad \beta = (2\,n)(3\,n{-}1)\cdots.$$

(a) Show that α generates a normal subgroup of D_{2n} of index 2.

(b) Show that every element of D_{2n} can be written in the form $\alpha^i \beta^j$ where $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1\}$.

(c) Show that every element in D_{2n} which does not lie in $\langle \alpha \rangle$ has order 2.

10. The *quaternion group* Q_8 of order 8 consists of eight elements

$$1, -1, i, -i, j, -j, k, -k$$

with multiplication given by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

(a) Show that Q_8 is generated by i and j.

(b) Show that $\langle i \rangle$ is a normal subgroup of Q_8 of index 2.

(c) Show that every element of Q_8 can be written as $i^m j^n$ where $m \in \{0, 1, 2, 3\}$ and $n \in \{0, 1\}$.

(d) Show that every element in Q_8 which does not lie in $\langle i \rangle$ has order 4.

(e) Show that Q_8 has a unique element of order 2.

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Problem Sheet II: Group Actions

1. (a) How many different ways can the cyclic group C_3 of order three act on the set $\{1, 2, 3, 4\}$?

(b) How many different ways can the cyclic group C_4 of order four act on the set $\{1, 2, 3\}$?

[Consider orbit decompositions and apply the Orbit-Stabiliser Theorem.]

2. (a) Let i_1, i_2, $\dots$, i_k be distinct points in $\Omega = \{1, 2, \dots, n\}$ and let σ be a permutation in S_n. By considering the effect on various points in Ω, or otherwise, show that

$$\sigma^{-1}(i_1\ i_2\ \dots\ i_k)\sigma = (i_1\sigma\ i_2\sigma\ \dots\ i_k\sigma).$$

Deduce that two permutations in S_n are conjugate if and only if they have the same cycle structure.

(b) Give a list of representatives for the conjugacy classes in S_5. How many elements are there in each conjugacy class? Hence calculate the order of and generators for the centralisers of these representatives.

3. Let G be a group and let Γ and Δ be sets such that G acts on Γ and on Δ. Define

$$(\gamma, \delta)^x = (\gamma^x, \delta^x)$$

for all $\gamma \in \Gamma$, $\delta \in \Delta$ and $x \in G$. Verify that this is an action of G on the set $\Gamma \times \Delta$.

Verify that the stabiliser of the pair (γ, δ) in this action equals the intersection of the stabilisers G_γ and G_δ (these being the stabilisers under the actions of G on Γ and Δ, respectively).

If G acts transitively on the non-empty set Ω, show that

$$\{\, (\omega, \omega) \mid \omega \in \Omega \,\}$$

is an orbit of G on $\Omega \times \Omega$. Deduce that G acts transitively on $\Omega \times \Omega$ if and only if $|\Omega| = 1$.

4. (a) There is a natural action of S_n on $\Omega = \{1, 2, \dots, n\}$. Show this action is transitive. How many orbits does S_n have on $\Omega \times \Omega$?

(b) Repeat part (a) with the action of the alternating group A_n on Ω.

5. Let G be a group and H be a subgroup of G. Show that the normaliser $\mathrm{N}_G(H)$ of H is the largest subgroup of G in which H is a normal subgroup.

[By *largest*, we mean that if L is any subgroup of G in which H is normal, then $L \leqslant \mathrm{N}_G(H)$. So you should check that (i) $H \trianglelefteqslant \mathrm{N}_G(H)$ and (ii) if $H \trianglelefteqslant L$ then $L \leqslant \mathrm{N}_G(H)$.]

6. Let G be a group and H be a subgroup of G. Let Ω be the set of right cosets of H in G. Define an action of G on Ω by

$$\begin{aligned} \Omega \times G &\to \Omega \\ (Hg, x) &\mapsto Hgx \end{aligned}$$

for $Hg \in \Omega$ and $x \in G$.

(a) Verify that this action is well-defined and that it is indeed a group action.

(b) Is the action transitive?

(c) Show that the stabiliser of the coset Hx is the conjugate H^x of H.

(d) Let $\rho\colon G \to \mathrm{Sym}(\Omega)$ be the permutation representation associated to the action of G on Ω. Show that

$$\ker \rho = \bigcap_{x \in G} H^x.$$

(e) Show that $\ker \rho$ is the largest normal subgroup of G contained in H.
[That is, show that (i) it is a normal subgroup of G contained in H and (ii) if K is any normal subgroup of G contained in H then $K \leqslant \ker \rho$.
This kernel is called the *core* of H in G and is denoted by $\mathrm{Core}_G(H)$.]

7. If H is a subgroup of G of index n, show that the index of the core of H in G divides $n!$.

8. Let G be a group and let G act on itself by conjugation. Show that the kernel of the associated permutation representation $\rho\colon G \to \mathrm{Sym}(G)$ is the centre $\mathrm{Z}(G)$ of G. Deduce that $\mathrm{Z}(G)$ is a normal subgroup of G.

9. Let G be a group and let $\mathrm{Aut}\, G$ denote the set of all automorphisms of G. Show that $\mathrm{Aut}\, G$ forms a group under composition.

For $g \in G$, let $\tau_g\colon G \to G$ be the map given by conjugation by g; that is,

$$\tau_g\colon x \mapsto g^{-1}xg \qquad \text{for all } x \in G.$$

Show that the map $\tau\colon g \mapsto \tau_g$ is a homomorphism $\tau\colon G \to \mathrm{Aut}\, G$. What is the kernel of τ?

Write $\mathrm{Inn}\, G$ for the image of τ. Thus $\mathrm{Inn}\, G$ is the set of *inner automorphisms* of G. Show that $\mathrm{Inn}\, G$ is a normal subgroup of $\mathrm{Aut}\, G$. [Hint: Calculate the effect of $\phi^{-1}\tau_g\phi$ on an element x, where $\phi \in \mathrm{Aut}\, G$, $\tau_g \in \mathrm{Inn}\, G$ and $x \in G$.]

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Problem Sheet III: Cauchy's Theorem and Sylow's Theorem

1. Let H be a subgroup of the symmetric group S_n of index 2. Show that $H = A_n$. [Hint: Show that H contains all squares of elements in S_n.]

2. Let G be a finite group, let p be a prime number and write $|G| = p^n m$ where p does not divide m. The purpose of this question is to use group actions to show G has a subgroup of order p^n; that is, G has a Sylow p-subgroup.

(a) Let Ω be the collection of all *subsets* of G of size p^n:

$$\Omega = \{ S \subseteq G \mid |S| = p^n \}.$$

Show that

$$|\Omega| = \binom{p^n m}{p^n} = m \left(\frac{p^n m - 1}{p^n - 1} \right) \left(\frac{p^n m - 2}{p^n - 2} \right) \dots \left(\frac{p^n m - p^n + 2}{2} \right) \left(\frac{p^n m - p^n + 1}{1} \right).$$

(b) Let j be an integer with $1 \leqslant j \leqslant p^n - 1$. Show that if the prime power p^i divides j, then p^i divides $p^n - j$. Conversely show that if p^i divides $p^n - j$, then p^i divides j.

Deduce that $|\Omega|$ is not divisible by p. [Examine each factor in the above formula and show that the power of p dividing the numerator coincides with the power dividing the denominator.]

(c) Show that we may define a group action of G on Ω by

$$\begin{aligned} \Omega \times G &\to \Omega \\ (S, x) &\mapsto Sx = \{ ax \mid a \in S \}. \end{aligned}$$

(d) Express Ω as a disjoint union of orbits:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_k.$$

Show that p does not divide $|\Omega_i|$ for some i.

(e) Let $S \in \Omega_i$ and let $P = G_S$, the stabiliser of S in the action of G on Ω. Show that p does not divide $|G : P|$ and deduce that p^n divides $|P|$.

(f) Fix $a_0 \in S$. Explain why $a_0 x \in S$ for all $x \in P$. Show that $x \mapsto a_0 x$ is an injective map $P \to S$. Deduce that $|P| \leqslant p^n$.

(g) Conclude that P is a Sylow p-subgroup of G.

3. Show that there is no simple group of order equal to each of the following numbers:

(i) 30;	(ii) 48;	(iii) 54;	(iv) 66;	(v) 72;
(vi) 84;	(vii) 104;	(viii) 132;	(ix) 150;	(x) 392.

[Note: These are not necessarily in increasing order of difficulty!]

4. Let G be a finite group, N be a normal subgroup of G and P be a Sylow p-subgroup of G.

(a) Show that $P \cap N$ is a Sylow p-subgroup of N.

(b) Show that PN/N is a Sylow p-subgroup of G/N.

[Hint: Show that the subgroup is of order a power of p and has index not divisible by p. In both parts expect to use the formula for the order of PN and the fact that P already has the required property as a subgroup of G.]

5. Let G be a finite group, p be a prime number dividing the order of G, and let P be a Sylow p-subgroup of G. Define

$$O_p(G) = \bigcap_{g \in G} P^g.$$

Show that $O_p(G)$ is the largest normal p-subgroup of G.

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Problem Sheet IV: Composition series and the Jordan–Hölder Theorem

1. Let G be a group and N be a normal subgroup of G. If G/N and N both have composition series, show that G has a composition series and that the set of composition factors of G equals the union of those of G/N and those of N.

[Use the Correspondence Theorem to lift the terms of a composition series to a chain of subgroups between G and N.]

2. Let p, q and r be prime numbers with $p < q < r$. Let G be a group of order pqr.

(a) Suppose that G does not have a unique Sylow r-subgroup. Show that it has a unique Sylow q-subgroup Q.
[Hint: How many elements of order r must there be and how many elements of order q if there were no normal Sylow q-subgroup?]

(b) Show that G/Q has a unique Sylow r-subgroup of the form K/Q where $K \trianglelefteqslant G$ and $|K| = qr$.

(c) Show that K has a unique Sylow r-subgroup and deduce that, in fact, G has a unique Sylow r-subgroup R (contrary to the assumption in (a)).
[Hint: Why is a Sylow r-subgroup of K also a Sylow r-subgroup of G? Remember that all the Sylow r-subgroups of G are conjugate.]

(d) Show that G/R has a unique Sylow q-subgroup.

(e) Deduce that G has a composition series

$$G = G_0 > G_1 > G_2 > G_3 = \mathbf{1}$$

where $|G_1| = qr$ and $|G_2| = r$. Up to isomorphism, what are the composition factors of G?

3. Let p be a prime number and let G be a non-trivial p-group. Show that G has a chain of subgroups

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

such that G_i is a normal subgroup of G and $|G : G_i| = p^i$ for $i = 0, 1, \ldots, n$.

What are the composition factors of G?

[Hint: Use the fact that $\mathrm{Z}(G) \neq 1$ to produce an element $x \in \mathrm{Z}(G)$ of order p. Consider the quotient group G/K where $K = \langle x \rangle$ and apply induction.]

4. The dihedral group D_8 has seven different composition series. Find all seven.

5. How many different composition series does the quaternion group Q_8 have?

6. Let G be a group, N be a normal subgroup of G and suppose that

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

is a composition series for G. Define $N_i = N \cap G_i$ for $i = 0, 1, \ldots, n$.

(a) Show that N_{i+1} is a normal subgroup of N_i for $i = 0, 1, \ldots, n-1$.

(b) Use the Second Isomorphism Theorem to show that

$$N_i/N_{i+1} \cong \frac{(G_i \cap N)G_{i+1}}{G_{i+1}}.$$

[Hint: Note that $N_{i+1} = G_{i+1} \cap (G_i \cap N)$.]

(c) Show that $(G_i \cap N)G_{i+1}$ is a normal subgroup of G_i containing G_{i+1}. Deduce that $(G_i \cap N)G_{i+1}/G_{i+1}$ is either equal to G_i/G_{i+1} or to the trivial group. [Remember G_i/G_{i+1} is simple.]

(d) Deduce that N possesses a composition series. [Hint: Delete repeats in the series (N_i).]

7. Let G be a group, N be a normal subgroup of G and suppose that

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

is a composition series for G. Define $Q_i = G_iN/N$ for $i = 0, 1, \ldots, n$.

(a) Show that Q_i is a subgroup of G/N such that Q_{i+1} is a normal subgroup of Q_i for $i = 0, 1, \ldots, n-1$.

(b) Use Dedekind's Modular Law to show that $G_{i+1}N \cap G_i = (G_i \cap N)G_{i+1}$. Show that

$$Q_i/Q_{i+1} \cong \frac{G_i/G_{i+1}}{(G_i \cap N)G_{i+1}/G_{i+1}}.$$

[Hint: Use the Third Isomorphism Theorem, the Second Isomorphism Theorem and note that $G_iN = G_i(G_{i+1}N)$ since $G_{i+1} \leqslant G_i$.]

(c) Show that $(G_i \cap N)G_{i+1}$ is a normal subgroup of G_i containing G_{i+1}. Deduce that $(G_i \cap N)G_{i+1}/G_{i+1}$ is either equal to G_i/G_{i+1} or to the trivial group. [Remember G_i/G_{i+1} is simple.]
Hence show that the quotient on the right hand side in (b) is either trivial or isomorphic to G_i/G_{i+1}.

(d) Deduce that G/N possesses a composition series.

8. The purpose of this question is to prove the Jordan–Hölder Theorem.

Let G be a group and suppose that

$$G = G_0 > G_1 > \cdots > G_n = \mathbf{1}$$

and

$$G = H_0 > H_1 > \cdots > H_m = \mathbf{1}.$$

Proceed by induction on n.

(a) If $n = 0$, observe that $G = \mathbf{1}$ and that the Jordan–Hölder Theorem holds in this (vacuous) case.

(b) Suppose $n \geqslant 1$. Consider the case when $G_1 = H_1$. Observe the conclusion of the theorem holds for this case.

(c) Now suppose $G_1 \neq H_1$. Use the fact that G_0/G_1 and H_0/H_1 are simple to show that $G_1 H_1 = G$. [Hint: Observe that it is a normal subgroup containing both G_1 and H_1.]

(d) Define $D = G_1 \cap H_1$. Show that D is a normal subgroup of G such that

$$G_0/G_1 \cong H_1/D \qquad \text{and} \qquad H_0/H_1 \cong G_1/D.$$

(e) Use Question 6 to see that D possesses a composition series

$$D = D_2 > D_3 > \cdots > D_r = \mathbf{1}.$$

(f) Observe that we now have four composition series for G:

$$\begin{aligned} G &= G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1} \\ G &= G_0 > G_1 > D = D_2 > \cdots > D_r = \mathbf{1} \\ G &= H_0 > H_1 > D = D_2 > \cdots > D_r = \mathbf{1} \\ G &= H_0 > H_1 > H_2 > \cdots > H_m = \mathbf{1}. \end{aligned}$$

Apply the case of part (b) (twice) and the isomorphisms of part (d) to complete the induction step of the proof.

9. Let G be a simple group of order 60.

(a) Show that G has no proper subgroup of index less than 5. Show that if G has a subgroup of index 5, then $G \cong A_5$.
[Hint for both parts: If H is a subgroup of index k, act on the set of cosets to produce a permutation representation. What do we know about the kernel?]

(b) Let S and T be distinct Sylow 2-subgroups of G. Show that if $x \in S \cap T$ then $|\mathrm{C}_G(x)| \geqslant 12$. Deduce that $S \cap T = \mathbf{1}$. [Hint: Why are S and T abelian?]

(c) Deduce that G has at most five Sylow 2-subgroups and hence that indeed $G \cong A_5$. [Hint: The number of Sylow 2-subgroups equals the index of a normaliser.]

Thus we have shown that there is a unique simple group of order 60 up to isomorphism.

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Problem Sheet V: Direct products and semidirect products

1. Give an example of two groups G and H and a subgroup of the direct product $G \times H$ which does not have the form $G_1 \times H_1$ where $G_1 \leqslant G$ and $H_1 \leqslant H$.

2. Let M and N be normal subgroups of a group G. By considering the map

$$g \mapsto (Mg, Ng),$$

or otherwise, show that $G/(M \cap N)$ is isomorphic to a subgroup of the direct product $G/M \times G/N$.

3. Using Question 2, or otherwise, show that if m and n are coprime integers, then $C_m \times C_n \cong C_{mn}$.

4. Let X_1, X_2, ..., X_n be non-abelian simple groups and let

$$G = X_1 \times X_2 \times \cdots \times X_n.$$

(In this question we shall identify the concepts of internal and external direct products. Thus we shall speak a subgroup of G containing a direct factor X_i when, if we were to use the external direct product notation strictly, we might refer to it containing the subgroup $\bar{X}_i$ in the notation of the lectures.)

Prove that a non-trivial normal subgroup of G necessarily contains one of the direct factors X_i. Hence show that every normal subgroup of G has the form

$$X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}$$

for some subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$.

[Hint: If N is a non-trivial normal subgroup of G, choose a non-identity element $(x_1, x_2, \dots, x_n) \in N$. Consider conjugating this element by the element $(1, \dots, 1, g, 1, \dots, 1)$.]

Now suppose X_1, X_2, ..., X_n are *abelian* simple groups. Is it still true that every normal subgroup of the direct product has this form?

5. Let p be a prime number.

(a) Show that $\operatorname{Aut} C_p \cong C_{p-1}$.

(b) Show that $\operatorname{Aut}(C_p \times C_p) \cong \mathrm{GL}_2(\mathbb{F}_p)$ (where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denotes the field of p elements).

[For (a), let $C_p = \langle x \rangle$. Observe that an automorphism α is given by $x \mapsto x^m$ where m is a representative for a non-zero element of $\mathbb{F}_p$. Recall the multiplication group of a finite field is cyclic.

For (b), write $C_p \times C_p$ additively and view it as a vector space over $\mathbb{F}_p$. Show that automorphisms of the group then correspond to invertible linear maps.]

6. Let $G = \langle x \rangle$ be a cyclic group. Show that $\operatorname{Aut} G$ is abelian.

7. Show that the dihedral group D_{2n} is isomorphic to a semidirect product of a cyclic group of order n by a cyclic group of order 2. What is the associated homomorphism $\phi\colon C_2 \to \operatorname{Aut} C_n$?

8. Show that the quaternion group Q_8 may not be decomposed (in a non-trivial way) as a semidirect product.

[Hint: How many elements of order 2 does Q_8 contain?]

9. Show that the symmetric group S_4 of degree 4 is isomorphic to a semidirect product of the Klein 4-group V_4 by the symmetric group S_3 of degree 3.

Show that S_4 is also isomorphic to a semidirect product of the alternating group A_4 by a cyclic group of order 2.

10. Let G be a group of order pq where p and q are primes with $p < q$.

(a) If p does not divide $q - 1$, show that $G \cong C_{pq}$, the cyclic group of order pq.

(b) If p does divide $q - 1$, show that there are essentially two different groups of order pq.

11. Classify the groups of order 52 up to isomorphism.

12. Show that a group of order 30 is isomorphic to one of

$$C_{30}, \quad C_3 \times D_{10}, \quad D_6 \times C_5, \quad D_{30}.$$

13. Classify the groups of order 98 up to isomorphism.

14. Classify the groups of order 117 up to isomorphism.

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Problem Sheet VI: Soluble groups

1. Let G and H be groups.

(a) Show that $(G \times H)' = G' \times H'$. Deduce that $(G \times H)^{(i)} = G^{(i)} \times H^{(i)}$ for all $i \geqslant 0$.

(b) Deduce that the direct product of two soluble groups is a soluble group. Could this result be deduced from other results proved in lectures?

2. Calculate the terms in the derived series for the following groups:

(i) S_3; (ii) D_8; (iii) A_4; (iv) S_4; (v) A_5; (vi) S_5.

[It may well help to remember that G' is the smallest normal subgroup of G such that G/G' is abelian.]

3. Let p be a prime number and let G be a finite p-group. Show that G is soluble.

[Hint: Proceed by induction, using the fact that $\mathrm{Z}(G) \neq \mathbf{1}$ for a non-trivial p-group G.]

4. Show that $\mathbf{1}$ and G are always characteristic subgroups of a group G. Give an example of a normal subgroup N of a group G such that N is not a characteristic subgroup of G.

5. Let G be a finite group and suppose that P is a Sylow p-subgroup of G (for some prime p) which is actually normal in G. Prove that P is a characteristic subgroup of G.

[Hint: Where does an automorphism map a Sylow p-subgroup?]

6. Let G be any group and let n be an integer. Define

$$G^n = \langle\, g^n \mid g \in G \,\rangle,$$

the subgroup of G generated by all its nth powers. Show that G^n is a characteristic subgroup of G. Show that every element in the quotient group G/G^n has order dividing n.

7. (a) Give an example of a group G with subgroups K and H such that $K \trianglelefteqslant H$, $H \trianglelefteqslant G$ but $K \ntrianglelefteq G$.

(b) Give an example of a group G with a characteristic subgroup H and a homomorphism $\phi\colon G \to K$ such that $H\phi$ is not a characteristic subgroup of $G\phi$.

(c) Give an example of a group G with subgroups H and L such that $H \leqslant L \leqslant G$, $H \operatorname{char} G$, but H is not a characteristic subgroup of L.

(d) Give an example of a group G with subgroups K and H such that $K \trianglelefteqslant H$, H is a characteristic subgroup of G but $K \ntrianglelefteq G$.

[Hint: A little thought should tell you that $|G| \geqslant 8$ is required for many of these examples. There are examples without making the groups considerably larger than this minimum.]

8. (a) Let M and N be normal subgroups of a group G that are both soluble. Show that MN is soluble.

[Hint: $M \trianglelefteqslant MN$. What can you say about the quotient MN/M?]

(b) Deduce that a finite group G has a largest normal subgroup S which is soluble.

['Largest' in the usual sense of containing all others. This normal subgroup S is called the *soluble radical* of G.]

(c) Prove that S is the unique normal subgroup of G such that S is soluble and G/S has no non-trivial abelian normal subgroup.

[Hint: To show G/S has no non-trivial abelian normal subgroup, remember that if H is a group with $N \trianglelefteqslant H$ such that H/N and N are soluble, then H is soluble. Use $N = S$ and choose H such that H/S is abelian.]

9. (a) Let M and N be normal subgroups of a group G such that G/M and G/N are both soluble. Show that $G/(M \cap N)$ is soluble.

[Hint: Use Question 2 on Problem Sheet V.]

(b) Deduce that a finite group G has a smallest normal subgroup R such that the quotient G/R is soluble.

['Smallest' in the usual sense of being contained in all others. This normal subgroup R is called the *soluble residual* of G.]

(c) Prove that R is the unique normal subgroup of G such that G/R is soluble and $R' = R$.

[Hint: To show $R' = R$, remember that if H is a group with $N \trianglelefteqslant H$ such that H/N and N are soluble, then H is soluble. Use $H = G/R'$ and $N = R/R'$.]

10. The following is an alternative way of proving that a minimal normal subgroup of a finite soluble group is an elementary abelian p-group.

Let G be a finite soluble group and M be a minimal normal subgroup of G.

(a) By considering M', prove that M is abelian. [Hint: $M' \operatorname{char} M$.]

(b) By considering a Sylow p-subgroup of M, prove that M is a p-group for some prime p.

(c) By considering the subgroup of M generated by all elements of order p, prove that M is an elementary abelian p-group.

11. Let G be the semidirect product of a cyclic group N of order 35 by a cyclic group C of order 4 where the generator of C acts by inverting the generator of N:

$$G = N \rtimes C = \langle\, x, y \mid x^{35} = y^4 = 1,\ y^{-1}xy = x^{-1}\,\rangle.$$

Find a Hall π-subgroup H of G, its normaliser $\mathrm{N}_G(H)$ and state how many Hall π-subgroups G possesses when (i) $\pi = \{2, 5\}$, (ii) $\pi = \{2, 7\}$, (iii) $\pi = \{3, 5\}$ and (iv) $\pi = \{5, 7\}$.

12. Let p, q and r be distinct primes with $p < q < r$. Let G be a group of order pqr.

(a) Show that G is soluble.

(b) Show that G has a unique Hall $\{q, r\}$-subgroup.

(c) How many Hall $\{p, r\}$-subgroups can G have? Can you construct examples to show that these numbers are indeed possible?

[Hints: Review Question 2 on Problem Sheet IV. Consider the normaliser of the Hall subgroup. Semidirect products are good ways to construct groups with normal subgroups.]

13. A *maximal subgroup* of G is a proper subgroup M of G such that there is no subgroup H strictly contained between M and G.

If p divides the order of the finite soluble group G, show that there is a maximal subgroup of G whose index is a power of p. [Hint: Consider Hall p'-subgroups.]

Show, by an example, that this is false for insoluble groups.

14. Let G be a finite soluble group whose order is divisible by k distinct prime numbers. Prove there is a prime p and a Hall p'-subgroup H such that $|G| \leqslant |H|^{k/(k-1)}$.

[Hint: Consider $|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ and consider the smallest value of $p_i^{n_i}$.]

15. The purpose of this question is to prove Theorem 6.33; that is, that any two Sylow bases in a finite soluble group G are conjugate.

Let G be a finite soluble group and let p_1, p_2, …, p_k be the distinct prime factors of G. Let $\mathscr{S}_i$ be the set of Hall p_i'-subgroups of G (for $i = 1, 2, \ldots, k$) and let $\mathscr{S}$ be the collection of all Sylow bases of G; that is, the elements of $\mathscr{S}$ are sequences (P_i) such that P_i is a Sylow p_i-subgroup of G for $i = 1, 2, \ldots, k$ and $P_i P_j = P_j P_i$ for all i and j.

(a) Show that

$$(Q_1, Q_2, \ldots, Q_k) \mapsto \Bigl(\bigcap_{j \neq 1} Q_j, \bigcap_{j \neq 2} Q_j, \ldots, \bigcap_{j \neq k} Q_j \Bigr)$$

is a bijection from $\mathscr{S}_1 \times \mathscr{S}_2 \times \cdots \times \mathscr{S}_k$ to $\mathscr{S}$.

(b) Now fix a representative Q_i in $\mathscr{S}_i$. Show that $|\mathscr{S}_i| = |G : \mathrm{N}_G(Q_i)|$. Deduce that $|\mathscr{S}_i|$ is a power of the prime p_i.

(c) Show that G acts on $\mathscr{S}$ according to the rule:

$$\bigl((P_i), g\bigr) \mapsto (P_i^g)$$

for $(P_i) \in \mathscr{S}$ and $g \in G$. (As usual, P_i^g denotes the conjugate of P_i by g.)

(d) Now concentrate on the specific Sylow basis (P_i) constructed from the Q_i as in lectures. [Also compare part (a).] Show that the stabiliser of (P_i) under the above action is the intersection $\bigcap_{i=1}^{k} \mathrm{N}_G(P_i)$ of the normalisers of the P_i, and that this coincides with the intersection $\bigcap_{j=1}^{k} \mathrm{N}_G(Q_j)$.

(e) Use part (b) to show that

$$\Bigl| G : \bigcap_{j=1}^{k} \mathrm{N}_G(Q_j) \Bigr| = \prod_{j=1}^{k} |G : \mathrm{N}_G(Q_j)|.$$

[Hint: Coprime indices!]

(f) Use part (a) to deduce that G acts transitively on $\mathscr{S}$.

MRQ 2009

School of Mathematics and Statistics

MT5824 Topics in Groups

Problem Sheet VII: Nilpotent groups

1. Show that $\gamma_2(G) = G'$. Deduce that abelian groups are nilpotent.

2. Show that $\mathrm{Z}(S_3) = \mathbf{1}$. Hence calculate the upper central series of S_3 and deduce that S_3 is not nilpotent.

Show that $\gamma_i(S_3) = A_3$ for all $i \geqslant 2$. [Hint: We have calculated S_3' previously and now know that S_3 is not nilpotent.]

Find a normal subgroup N of S_3 such that S_3/N and N are both nilpotent.

3. Show that $\mathrm{Z}(G \times H) = \mathrm{Z}(G) \times \mathrm{Z}(H)$.

Show, by induction on i, that $\mathrm{Z}_i(G \times H) = \mathrm{Z}_i(G) \times \mathrm{Z}_i(H)$ for all i.

Deduce that a direct product of a finite number of nilpotent groups is nilpotent.

4. Let G be an finite elementary abelian p-group. Show that $\Phi(G) = \mathbf{1}$.

5. Let G be a finite p-group.

If M is a maximal subgroup of G, show that $|G : M| = p$. [Hint: G is nilpotent, so $M \trianglelefteqslant G$.]

Deduce that $G^p G' \leqslant \Phi(G)$.

Use the previous question to show that $\Phi(G) = G^p G'$.

Show that G can be generated by precisely d elements if and only if $G/\Phi(G)$ is a direct product of d copies of the cyclic group of order p.

6. Let G be a nilpotent group with lower central series

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_c(G) > \gamma_{c+1}(G) = \mathbf{1}.$$

Suppose N is a non-trivial normal subgroup of G. Choose i to be the largest positive integer such that $N \cap \gamma_i(G) \neq \mathbf{1}$. Show that $[N \cap \gamma_i(G), G] = \mathbf{1}$.

Deduce that $N \cap \mathrm{Z}(G) \neq \mathbf{1}$.