School of Mathematics and Statistics

MT4003 Groups

Problem Sheet I: Definition and Examples of Groups

1. Consider the following two permutations from the symmetric group S_7 of degree 7:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 5 & 6 & 4 & 7 \end{pmatrix}, \qquad \beta = (2\,3\,4)(1\,2\,5)(3\,6\,1\,7).$$

   Write each of the following permutations as products of disjoint cycles: (i) α; (ii) β; (iii) $\alpha\beta$; (iv) $\beta\alpha$; (v) α^{-1}; (vi) β^{-1}; (vii) $(\alpha\beta)^{-1}$; (viii) $\beta^{-1}\alpha^{-1}$.

2. Show that the symmetric group S_n is non-abelian if and only if $n \geqslant 3$.

3. Consider $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, the field of order 5 where arithmetic is calculated modulo 5. You may assume that it is a field (and those who have covered MT2002 should already know this).

   Consider the general linear group $\mathrm{GL}_2(\mathbb{F}_5)$, more often denoted by $\mathrm{GL}_2(5)$ (or, in some books, by $\mathrm{GL}(2,5)$), which consists of all 2×2 matrices with entries from $\mathbb{F}_5$ and non-zero determinant.

   (a) For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$, compute AB and A^{-1}.

   (b) Prove that $\mathrm{GL}_2(5)$ is non-abelian.

   (c) Determine the order of $\mathrm{GL}_2(5)$.
   [Hint: A matrix is invertible if and only if its rows (or, equivalently, columns) are linearly independent. How do you choose a linearly independent pair of vectors from $\mathbb{F}_5^2$?]

4. Let F be an arbitrary field. Show that the general linear group $\mathrm{GL}_n(F)$ is non-abelian if and only if $n \geqslant 2$.

5. Verify that the Klein 4-group is indeed a group.
   [Do not check each of the 64 cases for associativity. Instead, use careful thought to reduce how many checks you need to perform.]

6. Consider the following three matrices with entries from the complex numbers:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   (a) Let $G = \{I, -I, A, -A, B, -B, C, -C\}$, where I denotes the usual 2×2 identity matrix. Calculate a multiplication table for G and hence show that matrix multiplication defines a binary operation on G.

   (b) Deduce that the quaternion group is indeed a group.

7. Let G be a group and a be an arbitrary (but fixed) element of G. Define a mapping $\tau_a \colon G \to G$ by $x\tau_a = a^{-1}xa$ for each $x \in G$. (This mapping is called *conjugation by* a.)

   (a) Specialise (for this part only) to the case $G = S_3$ and $a = (1\ 2\ 3)$. Compute $\sigma\tau_a$ for each $\sigma \in S_3$.

   (b) Prove that τ_a is always an isomorphism (in any group).

   (c) If G is an arbitrary group, find an element $a \in G$ such that τ_a is the identity mapping (that is, $x\tau_a = x$ for all $x \in G$).

   (d) Prove that G is abelian if and only if *all* the mappings τ_a are equal to the identity mapping.

8. Let G, H and K be groups.

   (a) If $\phi \colon G \to H$ is an isomorphism, show that its inverse $\phi^{-1} \colon H \to G$ is also an isomorphism.

   (b) If $\phi \colon G \to H$ and $\psi \colon H \to K$ are isomorphisms, show that $\phi\psi \colon G \to K$ is also an isomorphism.

   (c) Show that $\cong$ (being isomorphic) is an equivalence relation on the class of all groups.

9. Let G be a group and define a mapping $\phi \colon G \to G$ by $x\phi = x^{-1}$ for each $x \in G$.

   (a) Prove that ϕ^2 $(= \phi\phi)$ is the identity mapping.

   (b) Prove that ϕ is a bijection.

   (c) Prove that ϕ is an isomorphism if and only if G is abelian.

   (d) Suppose that G is a finite group of even order. Prove that there exists $a \in G$ of order 2.

10. Let G be any group. If $x \in G$, prove that $(x^{-1})^{-1} = x$.

11. Let G be a group and suppose that $x^2 = 1$ for all $x \in G$. Show that G is abelian.

12. Let G be a group and let $a, b \in G$. Prove that if $a^2 = 1$ and $b^2a = ab^3$, then $b^5 = 1$.
[Hint: $b^4 = b^2b^2aa$, $b^6 = b^2b^2b^2$.]

13. Let G be a group.

(a) Prove that for any $x, y \in G$, the following formulae involving the order of an element hold: $o(x) = o(x^{-1})$; $o(x) = o(y^{-1}xy)$, and $o(xy) = o(yx)$.

(b) If σ is a permutation in S_n decomposed as a product of disjoint cycles as $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$, where σ_i is a cycle of length r_i, then prove that

$$o(\sigma) = \operatorname{lcm}(r_1, r_2, \ldots, r_k).$$

(c) Consider the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

from the general linear group $\mathrm{GL}_2(\mathbb{Q})$. Show that $o(A) = 4$, $o(B) = 3$, but that AB has infinite order.

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet II: Subgroups

1. Consider $H = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Show that H is a subgroup of S_4. Construct the multiplication table of H. Is H isomorphic to a group already defined in the course?

2. Let $n > 1$. Show that $H = \{\, \sigma \in S_n \mid n\sigma = n \,\}$ is a subgroup of S_n isomorphic to S_{n-1}.

   [Hint: Define a suitable isomorphism $S_{n-1} \to H$. Also note in the above formula, $n\sigma$ denotes the image of n under the permutation σ.]

3. Let G be an abelian group and H be a subgroup of G. Show that H is also abelian.

4. Let G be a group and H be a subset of G. Show that H is a subgroup of G if and only if H is a non-empty subset of G such that $xy^{-1} \in H$ whenever $x, y \in H$.

5. Let G be a group and suppose that H and K are subgroups of G. Prove that the union $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.

6. Let a and b be elements of a group that commute (that is, such that $ab = ba$). Show that

   $$o(ab) \leqslant \operatorname{lcm}(o(a), o(b)).$$

   Deduce that the set of all elements of finite order in an abelian group is a subgroup. [This is called the *torsion subgroup*.]

   Is the same true for the set of elements of finite order in an arbitrary group?

7. Let n be a positive integer. Let

   $$C_n = \{1, \mathrm{e}^{2\pi i/n}, \mathrm{e}^{4\pi i/n}, \dots, \mathrm{e}^{2(n-1)\pi i/n}\},$$

   the set of all complex numbers z satisfying $z^n = 1$.

   (a) Show that C_n forms a group under multiplication.
   (b) Is C_n abelian?
   (c) Let $g = \mathrm{e}^{2\pi i/n}$. Show that $C_n = \langle g \rangle$.

8. Let $\sigma = (1\ 4)(2\ 6)$ and $\tau = (2\ 5)(3\ 6)$. Find all elements in the subgroup of S_6 generated by σ and τ.

9. Let n be a positive integer with $n \geqslant 3$. Let Δ denote the regular polygon with n sides. A *symmetry* of Δ is an invertible transformation that maps vertices to vertices and edges to edges.

   (a) How many symmetries of Δ are there? How many of these are rotations and how many are reflections?

   (b) Is it true that the composition of two symmetries is again a symmetry of Δ?

   (c) Show that the set G of all symmetries of Δ forms a group under composition.

   Label the vertices of Δ clockwise 1, 2, ..., n and set $X = \{1, 2, \dots, n\}$. For each $g \in G$, define a map $\phi_g \colon X \to X$ by:

   $k\phi_g$ is the label ℓ such that g maps the vertex labelled k to the vertex labelled ℓ.

   (d) Show that ϕ_g is a permutation of X for every $g \in G$.

   (e) Let a denote the rotation of Δ clockwise through an angle of $2\pi/n$ and let b denote the reflection of Δ in the axis that passes through the vertex 1. Calculate ϕ_a and ϕ_b.

   (f) Show that $g \mapsto \phi_g$ is an isomorphism $G \to D_{2n}$ from the symmetry group of Δ to the dihedral group of order $2n$.

10. (a) Find the order of the symmetric group of degree n.

    (b) Show that the alternating group A_n has index 2 in S_n and hence determine its order.

11. Consider the symmetric group S_4 of degree 4. For each divisor m of its order, find a subgroup of S_4 of order m.

12. For each of the following subgroups H of S_4, determine its index in S_4. Also find a system of representatives for the right cosets in S_4; that is, find a subset $T = \{t_1, t_2, \dots, t_k\}$ of S_4 such that the distinct right cosets of H in S_4 are Ht_1, Ht_2, ..., Ht_k:

    (a) $H = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$;

    (b) $H = S_3$;

    (c) $H = D_8$.

13. (a) List the elements of the dihedral group of order 8 and determine the order of each element.

    Is this dihedral group isomorphic to C_8? Is it isomorphic to the quaternion group Q_8?

    (b) Is the dihedral group of order 12 isomorphic to A_4?

    (c) Is the dihedral group of order 24 isomorphic to S_4?

14. Let G be a group and H be a subgroup of G. Show that two left cosets xH and yH (where $x, y \in G$) are equal if and only if $x^{-1}y \in H$.

15. Let G be a group and H and K be subgroups of G. Let $\mathcal{C}_H$, $\mathcal{C}_K$ and $\mathcal{C}_{H\cap K}$ denote the set of right cosets in G of H, K and $H \cap K$, respectively. Define a mapping $\alpha \colon \mathcal{C}_{H\cap K} \to \mathcal{C}_H \times \mathcal{C}_K$ by

    $$\big((H \cap K)x\big)\alpha = (Hx, Kx).$$

    Prove that α is well-defined and one-one.

    Deduce that

    $$|G : H \cap K| \leqslant |G : H| \cdot |G : K|.$$

    Deduce that the intersection of two subgroups of G of finite index is also a subgroup of finite index.

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet III: Normal Subgroups, Quotient Groups & Homomorphisms

1. Let G be a group and N be a subgroup of G. Show that the following conditions on N are equivalent:
   (a) N is a normal subgroup of G.
   (b) $g^{-1}Ng = N$ for all $g \in G$.
   (c) $Ng = gN$ for all $g \in G$.
   (d) Every right coset of N in G is also a left coset of N.

2. Let G be a group and N be a subgroup of index 2. Show that N is a normal subgroup of G.

3. Let G be a group and suppose M and N are normal subgroups of G. Show that $M \cap N$ is a normal subgroup of G and that the set of products

   $$MN = \{ \, xy \mid x \in M, y \in N \, \}$$

   is a normal subgroup of G.

4. Let σ be any permutation in S_n and let

   $$\tau = (i_1 \; i_2 \; \ldots \; i_r)$$

   be an r-cycle. Show that $\sigma^{-1}\tau\sigma$ is the r-cycle

   $$(i_1\sigma \;\; i_2\sigma \;\; \ldots \;\; i_r\sigma).$$

   If an arbitrary permutation τ is written as a product of disjoint cycles, what does this tell you about the conjugate $\sigma^{-1}\tau\sigma$?

5. Let G be a group, N be a normal subgroup of G and x be an element of G of finite order. Show that the element Nx of the quotient group G/N has finite order and that $o(Nx)$ divides $o(x)$.

6. Let N be the subgroup of S_4 generated by $(1\;2)(3\;4)$ and $(1\;3)(2\;4)$.
   (a) Find the elements of N and prove that N is normal in S_4.
   (b) List the elements of the quotient S_4/N. Is S_4/N cyclic? Is it abelian?

7. Let $G = S_3$ and $H = \langle (1\ 2) \rangle$.

   (a) Show that H is not a normal subgroup of S_3.

   (b) Show that $H(1\ 3) = H(1\ 2\ 3)$ and $H(2\ 3) = H(1\ 3\ 2)$.

   (c) Is $H(1\ 3)(1\ 3\ 2) = H(1\ 2\ 3)(2\ 3)$?

   (d) What do (b) and (c) illustrate?

8. Consider the quaternion group Q_8.

   (a) Find all orders of elements in Q_8.

   (b) Find all subgroups of Q_8.

   (c) Show that every subgroup of Q_8 is normal.

   (d) Describe, up to isomorphism, all homomorphic images of Q_8.

9. Show that the alternating group A_4 does not possess a subgroup of order 6.

   What does this tell you about Lagrange's Theorem?

10. (a) Give an example of a group G possessing two subgroups H and K with $K \leqslant H \leqslant G$, $K \trianglelefteq H$ and $H \trianglelefteq G$, but $K \ntrianglelefteq G$.

    (b) Give an example of a group G with two subgroups H and K such that the set of products
    $$HK = \{\, hk \mid h \in H, k \in K \,\}$$
    is not a subgroup of G.

    [Hint: Both these happen fairly often and you know some fairly small examples of groups where it does happen!]

11. Let G be a group and suppose that G has a chain of subgroups
    $$G = G_0 > G_1 > \cdots > G_n = \mathbf{1}$$
    with the property that each G_{i+1} is a normal subgroup of G_i and the quotient group G_i/G_{i+1} is abelian for all i.

    If H is any subgroup of G, show that H also has such a chain of subgroups with abelian quotients.

    [Hint: Consider the groups $H \cap G_i$ and use the Second Isomorphism Theorem.]

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet IV: Cyclic Groups

1. For each of the following orders n, describe the subgroup structure of the finite cyclic group of order n:

   (i) 12; (ii) 30; (iii) p^2q^2

   where p and q are distinct primes.

2. (a) Let G be a cyclic group and H be a subgroup of G. Show that G/H is also cyclic.
   (b) Now suppose G is a cyclic group of order n and that H is of order d. What is G/H?

3. Let $G = C_{12}$, a cyclic group of order 12.

   (a) For each divisor d of 12, list how many elements there are in G of order d.
   (b) How many elements x are there such that $G = \langle x \rangle$?
   (c) How many sets A are there with $|A| = 2$ and $G = \langle A \rangle$?

   [Hint: Draw the subgroup diagram for G and consider in which subgroups elements of particular order must belong.]

4. Let $G = C_{p^n}$, a cyclic group of order p^n.

   (a) For each $i = 0, 1, \dots, n$, list how many elements there are in G of order p^i.
   (b) How many elements x are there such that $G = \langle x \rangle$?
   (c) How many sets A are there with $|A| = 2$ and $G = \langle A \rangle$?

5. Let $G = \langle x \rangle$ be an infinite cyclic group.

   (a) Let H be a non-trivial subgroup of G. Show that H contains an element of the form x^k for some $k > 0$.
   (b) Show that $H = \langle x^m \rangle$ where m is the least positive integer such that $x^m \in H$.
   (c) Show that every coset of H has the form Hx^i where $i \in \{0, 1, \dots, m-1\}$. Show that $|G : H| = m$.
   (d) Deduce that, if m is a positive integer, $H = \langle x^m \rangle$ is the unique subgroup of G of index m.
   (e) Show that $G/H \cong \mathbb{Z}_m$.

   [Hint: For (ii), mimic the second half of the corresponding proof for finite cyclic groups. Use division, with quotient and remainder, in $\mathbb{Z}$.]

6. Let G be any group and N be a normal subgroup that is cyclic. Show that every subgroup of N is also normal in G.

   Is it true that if G is any group and N is an abelian normal subgroup of G, then every subgroup of N is also normal? If not, provide a counterexample.

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet V: Constructing Groups

1. A permutation group $G \leqslant S_X$ is called *transitive* if for all $x, y \in X$ there exists some $\sigma \in G$ such that $x\sigma = y$.

   (a) Prove that every group is isomorphic to a transitive subgroup of some symmetric group.

   (b) What is the smallest positive integer n such that a cyclic group C_6 of order 6 is isomorphic to a transitive subgroup of S_n?

   (c) What is the smallest positive integer n such that a cyclic group C_6 of order 6 is isomorphic to a subgroup of S_n?

2. Let n be a positive integer. If σ is an arbitrary permutation in S_n, define $\bar{\sigma} \in S_{n+2}$ by

$$\bar{\sigma} = \begin{cases} \sigma & \text{if } \sigma \text{ is even} \\ \sigma(n{+}1\ n{+}2) & \text{if } \sigma \text{ is odd.} \end{cases}$$

   Show that the map $\phi \colon S_n \to S_{n+2}$ by $\sigma\phi = \bar{\sigma}$ is an injective homomorphism with $\operatorname{im}\phi \leqslant A_{n+2}$.

   Deduce that every finite group is isomorphic to a subgroup of an alternating group.

3. Let F be a field and let $V = F^n$ be the vector space of row vectors with entries from F. Let $\mathscr{E} = \{\mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_n\}$ be the standard basis for V.

   (a) If σ is a permutation in S_n, define a linear map $T_\sigma \colon V \to V$ by

$$\mathrm{e}_i T_\sigma = \mathrm{e}_{i\sigma} \qquad \text{for } i = 1, 2, \dots, n.$$

   Show that $T_\sigma T_\tau = T_{\sigma\tau}$ for all $\sigma, \tau \in S_n$.

   (b) As we are writing maps on the right, the matrix T of a linear transformation with respect to $\mathscr{E}$ is obtained by expressing $\mathrm{e}_i T$ in terms of the basis and writing the coefficients along the *rows* of the matrix.

   Let A_σ denote the matrix of T_σ with respect to $\mathscr{E}$. Describe the entries of A_σ. (That is, specify which entries equal 1 and which equal 0.)

   (c) Show that the map $\phi \colon S_n \to \mathrm{GL}_n(F)$ by $\sigma \mapsto A_\sigma$ is an injective homomorphism.

   (d) Deduce that every finite group is isomorphic to a subgroup of $\mathrm{GL}_n(F)$ for some positive integer n.

4. Let G and H be any groups.

   (a) Show that $G \times H \cong H \times G$.

   (b) Show that $G \cong G \times \mathbf{1}$.

   (c) Show that $G \times H$ is abelian if and only if both G and H are abelian.

   (d) Show that if $g \in G$ and $h \in H$, then

$$o\big((g,h)\big) = \operatorname{lcm}(o(g), o(h)).$$

   [The left-hand side denotes the order of the element (g,h) in the direct product $G \times H$.]

5. Construct three non-abelian groups of order 24 that are pairwise non-isomorphic. Prove that they are indeed not isomorphic to each other.

6. (a) Show that Q_8 is directly indecomposable (that is, it cannot be written as $Q_8 \cong M \times N$ for two non-trivial groups M and N).

   (b) Show that a cyclic group C_{p^n} of prime-power order is directly indecomposable. [Hint: How many subgroups of order p does it possess?]

7. Consider the direct product $G = S_3 \times S_3$ of two copies of the symmetric group of degree 3.

   (a) Let $N = A_3 \times A_3$. Find all normal subgroups of G that are contained in N. [Hint: First describe all the subgroups of N. What is the conjugate of an element (x,y) by an element $(1,g)$?]

   (b) How many normal subgroups of G are there containing N?

   (c) Find all the normal subgroups of G and hence all quotient groups of G.

8. Let G_1, G_2, …, G_k be groups and let $D = G_1 \times G_2 \times \cdots \times G_k$ be their direct product.

   (a) Show that D is indeed a group.

   (b) Find a homomorphism from G_i with image $\bar{G}_i = \{\, (1,\dots,1,g,1,\dots,1) \mid g \in G_i \,\}$. Show that $\bar{G}_i$ is a normal subgroup of D isomorphic to G_i.

   (c) Describe the quotient group $D/\bar{G}_i$. Justify your answer.

   (d) Show that $D = \bar{G}_1 \bar{G}_2 \dots \bar{G}_k$.

   (e) Show that $\bar{G}_i \cap (\bar{G}_1 \dots \bar{G}_{i-1} \bar{G}_{i+1} \dots \bar{G}_k) = \mathbf{1}$ for all i.

9. Let G be a group possessing normal subgroups N_1, N_2, …, N_k such that $G = N_1 N_2 \dots N_k$ and $N_i \cap (N_1 \dots N_{i-1} N_{i+1} \dots N_k) = \mathbf{1}$ for all i. Show that

$$G \cong N_1 \times N_2 \times \cdots \times N_k.$$

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet VI: Finite Abelian Groups

1. List all the abelian groups of order 64 up to isomorphism.

   List all the abelian groups of order 2025 up to isomorphism.

2. List all the abelian groups of order 1000 up to isomorphism.

   For the group $G = C_8 \times C_5 \times C_{25}$, find the orders of elements and the number of elements of each order.

   Do there exist two non-isomorphic abelian groups of order 1000 with the same number of elements of order 10?

3. Prove that a finite abelian group that is not cyclic contains a subgroup isomorphic to $C_p \times C_p$ for some prime p.

   Prove that no cyclic group contains such a subgroup.

4. Prove that, for every finite abelian group G and every divisor m of $|G|$, there is a subgroup of G of order m.

   [Hint: First consider the case when G is cyclic of prime-power order.]

5. Let p be a prime and suppose that G is an abelian group of order a power of p. Write

   $$G \cong C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_t}}$$

   as a direct product of cyclic groups of prime-power order.

   Show that $M = \{ x \in G \mid x^p = 1 \}$ is a subgroup of G and that

   $$M \cong \underbrace{C_p \times C_p \times \cdots \times C_p}_{t \text{ times}}.$$

   Deduce that if H is a subgroup of G, then H is a direct product of at most t cyclic groups of prime-power order.

6. Consider the group $G = C_3 \times C_9 \times C_{81}$. Describe the subgroups of G up to isomorphism.

7. Consider the group $G = C_3 \times C_3$. Determine the number of ordered pairs (x, y) such that $\{x, y\}$ is a generating set for G.

8. Consider the group $G = C_3 \times C_3 \times C_{27}$.

   (a) Determine how many elements there are in G of order 3.

   (b) Determine how many elements there are in G of order 9.

   (c) Determine how many subgroups there are in G of order 3.

   (d) Determine how many subgroups there are in G of order 9. [Hint: Use Question 7.]

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet VII: Simple Groups

1. Show that any simple abelian group is cyclic of prime order.

2. Let $n \geqslant 5$. Show that the normal subgroups of the symmetric group of degree n are precisely $\mathbf{1}$, A_n and S_n.

   [Hint: If $N \trianglelefteq S_n$, consider $N \cap A_n$.]

3. (a) Let K be the subgroup of A_5 generated by its 5-cycles. Show that K is a normal subgroup of A_5 and hence deduce that $K = A_5$.

   (b) Suppose that H is a subgroup of S_5 such that $|S_5 : H| < 5$. Let σ be any 5-cycle. By considering cosets of H containing powers of σ, show that σ must lie in H.

   Hence prove that $H = A_5$ or S_5.

   (c) Show that A_5 has no proper subgroup of index less than 5.

   [Hint for (b): If a coset of H contains two distinct elements from $\{1, \sigma, \sigma^2, \sigma^3, \sigma^4\}$, show that H contains σ.]

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet VIII: The Centre, Commutators and Conjugation

1. Let G be a group and H be a subgroup of G such that $H \leqslant \mathrm{Z}(G)$. Show that H is a normal subgroup of G.

2. (a) Show that centre of S_n is trivial for $n \geqslant 3$.
   (b) Show that the centre of the dihedral group D_{2n} is trivial if n is odd and is cyclic of order 2 if n is even.
   (c) Show that the centre of A_n is trivial for $n \geqslant 4$.
   (d) What is $\mathrm{Z}(A_3)$?

3. Let p be a prime number. Show that
$$\mathrm{Z}(\mathrm{GL}_2(p)) = \{\, \alpha I \mid \alpha \in \mathbb{F}_p, \alpha \neq 0 \,\},$$
where I denotes the identity matrix.

4. Prove that if H is a subgroup of a group G such that $G' \leqslant H$, then H is a normal subgroup of G.

5. Let G and H be groups. Prove that $(G \times H)' = G' \times H'$.

   [Hint: Prove that every generator of each of these groups belongs to the other.]

6. Find A_n' and S_n' for each $n \geqslant 1$.

   [Hint: You should consider different possibilities for n. For example, $n \geqslant 5$ corresponds to different behaviour to $n = 4$, etc.]

7. List the conjugacy classes of the symmetric group S_5, giving one element from each conjugacy class and the size of each conjugacy class.

   What is the order of the centraliser $\mathrm{C}_{S_5}((1\ 2\ 3\ 4\ 5))$?

   How many conjugates does $(1\ 2\ 3\ 4\ 5)$ have in A_5?

   Is $(1\ 2\ 3\ 4\ 5)$ conjugate to $(1\ 2\ 3\ 5\ 4)$ in A_5?

8. Find the sizes of the conjugacy classes in S_4.

   Hence determine all normal subgroups of S_4.

9. Let n be odd and let α and β denote the usual generators of the dihedral group D_{2n} of order $2n$.

   Find the centralisers of the elements β and α^i (where $0 \leqslant i < n/2$).

   Hence determine the conjugacy classes of D_{2n} when n is odd.

   What are the conjugacy classes of D_{2n} when n is even?

10. Let G be a non-abelian simple group. Show that every conjugacy class of G, apart from $\{1\}$, contains at least two elements.

11. A *finite p-group* is a finite group whose order is a power of the prime number p.

    Use the Class Equation to show that if G is a non-trivial p-group, then the centre $\mathrm{Z}(G)$ of G is non-trivial.

    [Hint: If an element x_i lies in a conjugacy class of size > 1, then the centraliser $\mathrm{C}_G(x_i)$ is a proper subgroup of G. What can you say about its index?]

12. Let p be a prime number. Show that a finite p-group is simple if and only if it is cyclic of order p.

13. Let G be any group.

    (a) The set of all automorphisms of G is denoted by $\operatorname{Aut} G$. Show that $\operatorname{Aut} G$ is a subgroup of the symmetric group S_G.

    (b) The set of all inner automorphisms of G is denoted by $\operatorname{Inn} G$. Define a map $\rho\colon G \to \operatorname{Inn} G$ by
    $$g \mapsto \tau_g$$
    (where $\tau_g\colon x \mapsto g^{-1}xg$). Show that ρ is a homomorphism.
    Deduce that $\operatorname{Inn} G$ is a subgroup of $\operatorname{Aut} G$ and that $\operatorname{Inn} G \cong G/\mathrm{Z}(G)$.

    (c) If $\phi \in \operatorname{Aut} G$ and $g \in G$, show that
    $$\phi^{-1}\tau_g\phi = \tau_{g\phi}.$$
    Show that $\operatorname{Inn} G$ is a normal subgroup of $\operatorname{Aut} G$.

14. Let $\sigma = (1\ 2)$ and $\tau = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$ be elements of S_8 and let $G = \langle \sigma, \tau \rangle$.

    Let $H = \langle (1\ 2)(3\ 4)(5\ 6)(7\ 8) \rangle$. Show that H is a normal subgroup of G. [Hint: Show σ, τ lie in its normaliser.]

    Find a proper normal subgroup of G that strictly contains H.

15. Let G be a group and H and K be subgroups of G. If H and K are conjugate in G, show that their normalisers are also conjugate in G.

School of Mathematics and Statistics

MT4003 Groups

Problem Sheet IX: Sylow's Theorem

1. Show that there is no simple group of order equal to each of the following numbers:

   (i) 42; (ii) 45; (iii) 54; (iv) 66.

2. Show that there is no simple group of order equal to each of the following numbers:

   (i) 30; (ii) 80; (iii) 992

   [Part (i) is probably the hardest!]

3. Let $G = \mathrm{GL}_2(5)$, the general linear group of degree 2 over the field $\mathbb{F}_5$ of 5 elements.

   Find the orders of the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}$.

   Find a Sylow 5-subgroup and a Sylow 3-subgroup of G. [Hint: The order of $\mathrm{GL}_2(5)$ was calculated on Problem Sheet I.]

   Are either of these Sylow subgroups unique?

4. (a) Let P be a non-trivial p-group. Show that $\mathrm{Z}(P)$ contains an element of order p. [Hint: See Question 11 on Problem Sheet VIII.]

   (b) If P is a p-group with $|P| = p^n$, show that P has a subgroup of order p^i for each i with $0 \leqslant i \leqslant n$. [Hint: Use induction and part (a).]

   (c) Deduce that if G is a finite group with $|G| = p^n m$ where p does not divide m, then G has a subgroup of order p^i for $0 \leqslant i \leqslant n$.

5. Let G be a finite group, p be a prime number dividing the order of G and let P be a Sylow p-subgroup of G. Show that

$$\bigcap_{g \in G} g^{-1} P g$$

   is the largest normal p-subgroup of G.