School of Mathematics and Statistics

MT3503 Complex Analysis

Problem Sheet I: Complex numbers and the topology of the complex plane

1. Let $z = a + bi$ be a complex number with real part a and imaginary part b. Determine the real and imaginary parts of
$$w = \frac{z+2}{z+1}$$
(expressed in terms of a and b).

2. Express each of the following complex numbers in modulus-argument form:
$$\text{(i) } 2 + 2i\sqrt{3}, \qquad \text{(ii) } -5 + 5i, \qquad \text{(iii) } \sqrt{3} - i.$$

3. Express the complex number $z = 1 - i$ in modulus-argument form and show that
$$(1 - i)^{16} = 256.$$

4. Let n be a positive integer.
   (a) Find all solutions (for complex numbers z) of the equation $z^n = 1$.
   (b) Find all solutions (for complex numbers z) of the equation $z^n = i$.

5. Find all solutions (for complex numbers z) of the equation $z^4 + z^2 + 1 = 0$.
   [Hint: Multiply by $z^2 - 1$.]

6. Show that if z is any complex number then
$$|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z| \leqslant \sqrt{2}|z|.$$
Give examples of complex numbers z illustrating that either inequality can be a strict inequality or equality.

7. Sketch the following subsets of the complex plane. Describe them (as clearly as possible) geometrically.
   (a) $A = \{\, z \in \mathbb{C} \mid |z - 1 - i| = 1 \,\}$
   (b) $B = \{\, z \in \mathbb{C} \mid |z - 1 + i| \geqslant |z - 1 - i| \,\}$
   (c) $C = \{\, z \in \mathbb{C} \mid |z + i| \neq |z - i| \,\}$
   (d) $D = \{\, z \in \mathbb{C} \mid \frac{1}{4}\pi < \operatorname{Arg} z \leqslant \frac{3}{4}\pi \,\}$
   (e) $E = \{\, z \in \mathbb{C} \mid \operatorname{Re} z < 1 \text{ or } \operatorname{Im} z \neq 0 \,\}$
   (f) $F = \{\, z \in \mathbb{C} \mid |z - 1| < 1 \text{ and } |z| = |z - 2| \,\}$

   [Hint: Recall that $|z - w|$ is the distance between the complex numbers z and w. The purpose of this question is to help enhance your geometric intuition when working with subsets of the complex plane. The method is to interpret what the condition defining the set is saying about the location of the complex number z.]

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Problem Sheet II: Limits, holomorphic functions, the Cauchy–Riemann equations, and power series

1. Define a function $f\colon \mathbb{C} \to \mathbb{C}$ by
$$f(x+iy) = 2x + iy$$
for $x, y \in \mathbb{R}$. Show that f is not differentiable at any point of $\mathbb{C}$.

2. Let a, b, c and d be complex numbers such that $ad - bc \neq 0$. Define a function f by
$$f(z) = \frac{az+b}{cz+d}.$$
A function of this form is called a *Möbius transformation.*

   (a) Show that f is holomorphic on the set $\mathbb{C} \setminus \{-d/c\}$ (or on the whole complex plane $\mathbb{C}$ if $c = 0$) and calculate its derivative.

   (b) By solving the equation $f(z) = w$, or otherwise, show that f is invertible on $\mathbb{C}\setminus\{-d/c\}$ and calculate the inverse.

   (c) Show that the composite of two Möbius transformations is again a Möbius transformation (that is, if f and g are two Möbius transformations, show that $f \circ g$ is also Möbius transformation).
   What is the link to matrix multiplication?

   (d) Calculate the matrix product
   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
   Let A be a 2×2 invertible matrix with complex number entries. Show that A can be expressed as a product involving the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and matrices of the form
   $$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$
   where $k, \lambda, \mu \in \mathbb{C}$ with $\lambda, \mu \neq 0$. [Hint: Consider elementary row operations.]

   (e) Show that every Möbius transformation can be expressed as the composite of *translation maps* (those of the form $z \mapsto z + k$ for some $k \in \mathbb{C}$), *inversion maps* ($z \mapsto 1/z$), *scaling maps* (those of the form $z \mapsto hz$ for some real number $h > 0$), a *rotation maps* (those of the form $z \mapsto \mathrm{e}^{i\theta} z$ for some θ).
   [Hint: Use (c) and (d).]

(f) Show that a Möbius transformation maps any circle or a line to another circle or a line.

[Hint: Use (e). It may help to describe a circle using an equation of the form $|z-c| = r$ and a line using an equation of the form $|z-a| = |z-b|$.]

3. Find the radius of convergence of the following power series:

$$\text{(i)}\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n, \qquad \text{(ii)}\ \sum_{n=0}^{\infty} z^n, \qquad \text{(iii)}\ \sum_{n=0}^{\infty} z^{5n},$$

$$\text{(iv)}\ \sum_{n=1}^{\infty} \frac{1}{n^n} z^n, \qquad \text{(v)}\ \sum_{n=0}^{\infty} n^n z^n.$$

4. Provide a function $f(z)$ that is holomorphic on the set $\mathbb{C} \setminus \{1\}$ and that coincides with the power series $\sum_{n=0}^{\infty} z^n$ inside its radius of convergence.

5. Let z be any complex number. Show that

(a) $\mathrm{e}^z \mathrm{e}^{-z} = 1$,

(b) $\mathrm{e}^z \neq 0$,

(c) $\overline{\mathrm{e}^z} = \mathrm{e}^{\bar{z}}$,

(d) $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re} z}$.

[If possible, avoid using the formula for $\mathrm{e}^{i\theta}$, for real θ, in the last part. You should be able to use part (c) to establish (d).]

6. Use the formulae for $\sin z$ in terms of e^{iz} and e^{-iz} to determine the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $\sin(x + iy)$ for real x and y.

Show that the solutions of $\sin z = 0$ are all real.

7. If z is any complex number, show that

$$\cosh z = \cos iz \qquad \text{and} \qquad \sinh z = -i \sin iz.$$

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Problem Sheet III: Contour integration & Cauchy's Theorem

1. Evaluate the integral
$$\int_\gamma (z^2 + 3z)\,\mathrm{d}z$$
where (i) γ is the line segment from 0 and $1+i$, and (ii) γ is the curve $\gamma(t) = t + it^2$ from 0 to $1+i$.

2. Evaluate the integral
$$\int_\gamma \frac{1}{z}\,\mathrm{d}z$$
where γ is the semi-circular arc given by $\gamma(t) = 4\mathrm{e}^{it}$ for $-\pi/2 \leqslant t \leqslant \pi/2$.

3. Let γ be the square contour with vertices at 0, π, $\pi + i\pi$ and $i\pi$. By integrating along each side in turn, verify that
$$\int_\gamma \sin z\,\mathrm{d}z = 0.$$
[This is to verify a special case of the general result, so *do not use Cauchy's Theorem*!]

4. Evaluate each of
$$\text{(a)} \quad \int_\gamma |z|^4\,\mathrm{d}z, \qquad \text{(b)} \quad \int_\gamma (\operatorname{Re} z)^2\,\mathrm{d}z,$$
$$\text{(c)} \quad \int_\gamma \frac{z^4 - 1}{z^2}\,\mathrm{d}z, \qquad \text{(d)} \quad \int_\gamma \sin z\,\mathrm{d}z$$
where (i) γ is the line segment from $-1+i$ to $1+i$, and (ii) γ is the circular contour of radius 1 about 0.

[You may use the Fundamental Theorem of Calculus for Integrals along a Curve if it applies.]

5. Using the theory developed so far, explain why each of the following integrals is zero *without performing the full calculation*:

   (a) $\displaystyle\int_\gamma \frac{1}{z-2}\,\mathrm{d}z$, where γ is a contour contained inside the open disc of radius 1 about 0.

   (b) $\displaystyle\int_\gamma \frac{\sin z}{z}\,\mathrm{d}z$, where γ is a circular contour of radius 2 about 1.
   [Hint: How have we defined $\sin z$ in this course?]

   (c) $\displaystyle\int_\gamma z|z|^4\,\mathrm{d}z$, where γ is a circular contour of radius 1 about 0.

   (d) $\displaystyle\int_\gamma \frac{1}{1+\mathrm{e}^z}\,\mathrm{d}z$, where γ is a circular contour of radius 1 about 1.

   [You may, and indeed should, use Cauchy's Theorem.]

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Problem Sheet IV: Consequences of Cauchy's Theorem: use of Cauchy's Integral Formula, Liouville's Theorem, Cauchy's Formula for Derivatives and Taylor's Theorem

1. Evaluate the following contour integrals:

   (a) $\displaystyle\int_\gamma \frac{1}{(z^2-1)(z^2-z-6)}\,\mathrm{d}z$, where γ is the positively oriented circular contour centred at 3 with radius 1.

   (b) $\displaystyle\int_\gamma \frac{1}{1+z^2}\,\mathrm{d}z$, where γ is the positively oriented circular contour centred at i with radius $r<2$.

   (c) $\displaystyle\int_\gamma \frac{\sin 3z}{z+\frac{1}{2}\pi}\,\mathrm{d}z$, where γ is the positively oriented circular contour centred at 0 with radius 5.

2. Let γ be a contour that does not pass through 0 or 1. Carefully determine all possible values of the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{e}^z}{z(z-1)}\,\mathrm{d}z.$$

3. (a) Evaluate the contour integral

$$\int_\gamma \frac{1}{z^2-3z+1}\,\mathrm{d}z$$

   where γ is the positively oriented circular contour of radius 1 about 0.

   (b) If $z=\mathrm{e}^{i\theta}$, show that

$$3-2\cos\theta = -\frac{z^2-3z+1}{z}.$$

   (c) Hence show that

$$\int_0^{2\pi}\frac{1}{3-2\cos\theta}\,\mathrm{d}\theta = \frac{2\pi}{\sqrt{5}}.$$

4. Suppose f is holomorphic on $\mathbb{C}$ and satisfies

$$f(z+2\pi)=f(z+2\pi i)=f(z) \qquad \text{for all } z\in\mathbb{C}.$$

   Show that f is constant.

5. Show that

$$\int_\gamma \frac{\mathrm{e}^{3z}}{(z+2)^4}\,\mathrm{d}z = \frac{9\pi i}{\mathrm{e}^6}$$

   where γ is the positively oriented circular contour of radius 4 about 0.

6. Let f be a function that is holomorphic on an open disc $B(a, r)$.

   (a) Suppose that $|f|$ is constant on $B(a, r)$. Show that f is also constant.

   [Hint: Express $|f|$ in terms of the real and imaginary parts of f. Make use of the Cauchy–Riemann Equations.]

   (b) Now suppose that $|f(z)| \leqslant |f(a)|$ for all $z \in B(a, r)$. Show that $f(z)$ is constant on $B(a, r)$.

   [Hint: Apply Cauchy's Integral Formula to express $f(a)$ in terms of a contour integral and then bound this integral. The purpose of this part is to establish a special case of what is known as the **Maximum-Modulus Theorem**.]

7. Consider the function $f(z) = z/(2 - z)$.

   What is the largest value $r > 0$ such that f is holomorphic on the open disc $B(0, r)$ of radius r about 0?

   Determine a power series expansion for f that is valid on the disc $B(0, r)$ for this value of r.

8. Suppose that f is holomorphic in $\mathbb{C}$, that M is a positive constant, and that m is a positive integer such that $|f(z)| \leqslant M|z|^m$ for all $z \in \mathbb{C}$. Using the formula for the coefficients in the Taylor series for f, or otherwise, show that $f(z)$ is a polynomial function of degree at most m.

9. Suppose that f is holomorphic on an open disc $B(a, r)$ and that $f(a) = 0$. Show that either f is identically zero on $B(a, r)$ or there exists some ε with $0 < \varepsilon < r$ such that f is non-zero on the set $B'(a, \delta) = \{\, z \in \mathbb{C} \mid 0 < |z - a| < \delta \,\}$.

   [Hint: Let $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ be the Taylor series valid in $B(a, r)$. If $f(z) \not\equiv 0$, there exists some smallest m with $c_m \neq 0$. Now consider the function $g(z) = \sum_{n=0}^{\infty} c_{m+n} (z-a)^n$.

   The purpose of this question is to establish a special case of what is known as the **Identity Theorem**. What is asked above is actually the main step in proving that theorem.]

10. (a) Suppose that $f \colon B(a, r) \to \mathbb{C}$ is continuous on the open disc $B(a, r)$ for some $a \in \mathbb{C}$ with $r > 0$. Assume that

   $$\int_\gamma f(z)\, \mathrm{d}z = 0$$

   for any triangular contour γ contained in $B(a, r)$. Define a function $F \colon B(a, r) \to \mathbb{C}$ by

   $$F(z) = \int_{[a,z]} f(w)\, \mathrm{d}w$$

   where $[a, z]$ denotes the line segment from a to z. Show that F is holomorphic on $B(a, r)$ with derivative f.

   (b) Deduce that if f is holomorphic on the open disc $B(a, r)$ then there exists some holomorphic function on $B(a, r)$ with derivative equal to f.

   (c) Deduce **Morera's Theorem**: If f is a continuous function on an open set U such that $\int_\gamma f(z)\, \mathrm{d}z = 0$ for all triangular contours contained in U, then f is holomorphic on U.

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Problem Sheet V: Harmonic Functions

1. Define a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ by
$$u(x,y) = 2x(1-y).$$
Show that u is harmonic on $\mathbb{R}^2$, find the harmonic conjugate of u and find a holomorphic function $f(z)$ on $\mathbb{C}$ such that $u(x,y) = \operatorname{Re} f(x+iy)$ for all $(x,y) \in \mathbb{R}^2$.

2. Define a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ by
$$u(x,y) = x\mathrm{e}^x \cos y - y\mathrm{e}^x \sin y.$$
Show that u is harmonic on $\mathbb{R}^2$, find the harmonic conjugate of u and find a holomorphic function $f(z)$ on $\mathbb{C}$ such that $u(x,y) = \operatorname{Re} f(x+iy)$ for all $(x,y) \in \mathbb{R}^2$.

3. Consider the function $u\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ given by
$$u(x,y) = x - \frac{y}{x^2+y^2}.$$
Show that u is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$, find the harmonic conjugate of u and find a holomorphic function $f(z)$ on $\mathbb{C} \setminus \{0\}$ such that $u(x,y) = \operatorname{Re} f(x+iy)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

4. Consider the function $u\colon \mathbb{R}^2 \to \mathbb{R}$ given by
$$u(x,y) = \sin(x^2-y^2)\,\mathrm{e}^{-2xy}.$$
Show that u is harmonic on $\mathbb{R}^2$, find the harmonic conjugate of u and find a holomorphic function $f(z)$ on $\mathbb{C}$ such that $u(x,y) = \operatorname{Re} f(x+iy)$ for all $(x,y) \in \mathbb{R}^2$.

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Problem Sheet VI: Singularities, Poles & Residues

1. Determine the coefficients of the terms z^n, for $-2 \leqslant n \leqslant 2$, in the Laurent series expansion of
$$f(z) = \frac{5}{z^2 - 3z - 4}$$
valid in the following three open subsets of $\mathbb{C}$:
   (a) $A = \{ z \in \mathbb{C} \mid |z| < 1 \}$;
   (b) $B = \{ z \in \mathbb{C} \mid 1 < |z| < 4 \}$;
   (c) $C = \{ z \in \mathbb{C} \mid |z| > 4 \}$.

   [Hint: First express $f(z)$ using partial fractions and then exploit the formula for the sum of a geometric series.]

2. Determine the Laurent series about $z = 1$ of
$$f(z) = \frac{1}{z(z-1)}$$
valid in the open annulus $A = \{ z \in \mathbb{C} \mid 0 < |z - 1| < 1 \}$.

3. Locate and classify the isolated singularities of the following functions:
$$\text{(i)}\ \frac{\mathrm{e}^z - 1}{z}, \qquad \text{(ii)}\ \frac{1}{\sin z}, \qquad \text{(iii)}\ z \sin(1/z).$$

4. Locate the poles in $\mathbb{C}$ of each of the following functions and determine the residue at each pole:
$$\text{(i)}\ \frac{1}{z^3(z^2+1)}, \qquad \text{(ii)}\ \frac{\mathrm{e}^z}{1-z}, \qquad \text{(iii)}\ \frac{\mathrm{e}^z}{z(z^2-1)},$$
$$\text{(iv)}\ \frac{1}{\sin z}, \qquad \text{(v)}\ \frac{1}{1-\mathrm{e}^z}.$$

5. Suppose that the function f has an isolated singularity at a. If f is holomorphic *and bounded* on the punctured open disc $B'(a, r)$, for some $r > 0$, show that f has a removable singularity at a.

   [Hint: Use the formula for the coefficients in the Laurent series given in Laurent's Theorem.]

6. Let f be a holomorphic function on an open set U containing a point c. Suppose that $f'(c) \neq 0$ and that there exists $r > 0$ such that $f(z) \neq f(c)$ for $0 < |z - c| < r$. Use Cauchy's Residue Theorem to show

$$\int_\gamma \frac{1}{f(z) - f(c)} \, \mathrm{d}z = \frac{2\pi i}{f'(c)}$$

whenever γ is a positively oriented circular contour about c of radius less than r.

[The hypothesis that such an r exists is actually unnecessary. It follows from the Identity Theorem (found in Priestley's book, but not covered, due to time constraints, in this module) that such an r always exists if $f'(c) \neq 0$.]

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Problem Sheet VII: Applications of contour integration

1. Show that

   (a)
$$\int_0^\infty \frac{1}{(x^2+4)(x^2+1)^2}\,\mathrm{d}x = \frac{\pi}{18}$$

   (b)
$$\int_{-\infty}^\infty \frac{1}{(x^2+x+1)^2}\,\mathrm{d}x = \frac{4\pi}{3\sqrt{3}}.$$

   (c)
$$\int_0^\infty \frac{x^3\sin x}{(x^2+1)^2}\,\mathrm{d}x = \frac{\pi}{4\mathrm{e}}.$$

   (d)
$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,\mathrm{d}x = \frac{(b-a)\pi}{2} \qquad \text{(for } a, b > 0\text{)}.$$

2. Evaluate the following integrals:

   (a)
$$\int_0^\infty \frac{x^2}{x^4+1}\,\mathrm{d}x.$$

   (b)
$$\int_0^\infty \frac{x\sin x}{x^2+4}\,\mathrm{d}x.$$

   (c)
$$\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,\mathrm{d}x \qquad \text{(for } a, b > 0\text{)}.$$

3. Evaluate the following integrals:

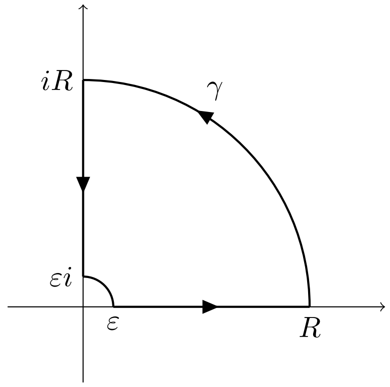
(i)
$$\int_0^{2\pi} \frac{1}{10+6\cos\theta}\,\mathrm{d}\theta$$

(ii)
$$\int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\,\mathrm{d}\theta$$

(iii)
$$\int_0^{2\pi} \frac{1}{(4\cos\theta-5)^2}\,\mathrm{d}\theta,$$

(iv)
$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,\mathrm{d}\theta.$$

4. Using the function
$$f(z) = \frac{e^{-z+iz}}{z}$$
and the contour γ shown below (where $0 < \varepsilon < R$),



show that
$$\int_0^\infty \frac{e^{-x}\sin x}{x}\,dx = \frac{\pi}{4}.$$

5. If z lies on the square contour that has vertices at the points $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$, show that
$$|\sin z| \geqslant 1.$$

6. By integrating the function
$$f(z) = \frac{1}{z^2 \sin z}$$
around the square contour that has vertices at the points $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$, show that
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

7. By integrating the function
$$f(z) = \frac{\cot \pi z}{4z^2+1}$$
around a suitable square contour, show that
$$\sum_{n=0}^{\infty} \frac{1}{4n^2+1} = \frac{1}{2} + \frac{\pi}{4}\coth\left(\frac{\pi}{2}\right).$$

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Problem Sheet VIII: Logarithms and related multifunctions

1. Show that
$$\int_0^\infty \frac{\log x}{x^4+1}\,\mathrm{d}x = -\frac{\pi^2}{8\sqrt{2}}.$$

2. With use of a suitable branch cut, define a holomorphic function $f(z)$ that is a suitable branch of the complex logarithm $\log(1+z)$ and such that f is holomorphic on the open disc $B(0,1)$ of radius 1 about 0.

   Using Taylor's Theorem, or otherwise, find a power series for $f(z)$ valid on the open disc $B(0,1)$.

3. Using contour integration, evaluate the integral
$$\int_0^\infty \frac{x^2\log x}{(1+x^2)^2}\,\mathrm{d}x.$$

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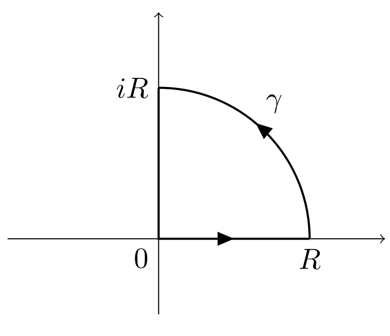
Problem Sheet IX: Locating and counting zeros and poles: Rouché's Theorem and the Argument Principle

1. Consider the equation
$$z^3 + 3z^2 + 2 = 0.$$
   (a) Show that this equation has two roots in the open disc $B = \{\, z \in \mathbb{C} \mid |z| < 1 + \sqrt{3} \,\}$.
   (b) Show that the third root of this equation lies in the annulus $A = \{\, z \in \mathbb{C} \mid 1 + \sqrt{3} \leqslant |z| < \sqrt{11} \,\}$.

2. Consider the equation
$$z^7 + 10z^4 + \tfrac{8}{9}z^3 + 8 = 0.$$
   (a) Determine the number of roots in the unit disc $B = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$.
   (b) Determine the number of roots in the annulus $A = \{\, z \in \mathbb{C} \mid 1 \leqslant |z| \leqslant 2 \,\}$.
   (c) Determine the number of roots in the annulus $C = \{\, z \in \mathbb{C} \mid 2 < |z| < \sqrt{5} \,\}$.

3. Consider the quadratic equation
$$z^2 - 2z + 4 = 0.$$
   (a) Use elementary methods to determine the roots of this equation.
   (b) Now use the Argument Principle to determine the number of roots of this equation in the first quadrant, by using the contour γ shown below (for some large value of R) and (i) showing the change of $\phi = \arg f(z)$ on the line segment $[0, R]$ is 0, (ii) the change of ϕ on the quarter circle is approximately π, and (iii) the change of ϕ as z travels from iR to 0 on the imaginary axis is approximately π.
   (c) Check that your answer to (b) agrees with your answer to (a).



4. Consider the equation
$$z^4 + iz^2 + 1 = 0.$$
   (a) Show that this equation has no real roots and no purely imaginary roots (i.e., no roots on the imaginary axis).
   (b) Using the Argument Principle, show that there is exactly one root in the first quadrant of the complex plane.
5. Consider the equation
$$8z^4 - 6z + 5 = 0.$$
   (a) Show that the equation has no real roots.
   (b) Show that the equation has no purely imaginary roots.
   (c) Show that there is one root in the first quadrant of the complex plane.
   (d) Show that there is one root in each quadrant of the complex plane.

   [Hint for (d): Consider complex conjugates.]