School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet VII: The adjoint of a linear transformation

1. Let V be a finite-dimensional inner product space (over either $\mathbb{R}$ or $\mathbb{C}$), let $S, T \colon V \to V$ be linear transformations and let α be a scalar. Using the definition of the adjoint, show that

   (a) $(S + T)^* = S^* + T^*$;

   (b) $(ST)^* = T^*S^*$;

   (c) $(\alpha T)^* = \bar{\alpha}T^*$;

   (d) $(T^*)^* = T$;

   (e) TT^* and T^*T are self-adjoint transformations.

   Are the transformations TT^* and T^*T necessarily equal? If not, give an example to show they can be different.

2. Let V and W be finite-dimensional inner product spaces and let $T \colon V \to W$ be a linear transformation. We shall define the *adjoint* of T to be a linear map $T^* \colon W \to V$ such that

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle$$

   for all $v \in V$ and $w \in W$.

   (a) Show that if the adjoint T^* exists, then it is unique.

   (b) Show that the adjoint of T does always exist.

   [Hint: Copy the proof of Lemma 7.2 as much as you can. This time use two orthonormal bases, one for V and one for W.]

3. Let V and W be finite-dimensional inner product space and $T \colon V \to W$ be an injective linear map. Show that

   (a) the linear map $T^*T \colon V \to V$ is invertible (where T^* is as defined in Question 2);

   (b) the linear map $P = T(T^*T)^{-1}T^*$ is the projection map onto the image $U = \operatorname{im} T$ associated to the direct sum decomposition $W = U \oplus U^{\perp}$.

   [Hint: Copy the proof of Proposition 6.19 as much as you can.]

4. Let V be a finite-dimensional inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$).

(a) If $w \in V$, show that $f_w \colon V \to F$, given by $f_w(v) = \langle v, w \rangle$ for $v \in V$, is a linear functional.

(b) Show that every linear function $f \colon V \to F$ is equal to f_w for some $w \in V$.
[Hint: Pick an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for V and consider the scalars $f(e_i)$.]

(c) Show that the map $\phi \colon V \to V^*$ given by $w \mapsto f_w$ is a bijection.
If $F = \mathbb{R}$, show that ϕ is an isomorphism.

(d) If $F = \mathbb{C}$, is it true that ϕ is an isomorphism?

Now fix a linear transformation $V \to V$.

(e) Show that the map $T' \colon V^* \to V^*$, given by $(T'f)(v) = f(T(v))$ for all $f \in V^*$ and $v \in V$, is a linear transformation of the dual space V^*.

(f) Show that $T'\phi = \phi T^*$, where T^* is the adjoint of T.

[Since ϕ is a bijection, the last part shows that the linear map induced on the dual space by T and the adjoint of T are closely related. They are often both denoted by T^*.]