School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet VI: Inner product spaces

1. Let $C[0,1]$ be the vector space of all real-valued continuous functions $f\colon [0,1] \to \mathbb{R}$. Verify that the following formula defines an inner product on $C[0,1]$:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

[Hint: It may help to use the fact that if $f(x)$ is a continous function such that $f(x) \geqslant 0$ for all x and $f(a) \neq 0$ for some $a \in [0,1]$, then $\int_0^1 f(x)\,\mathrm{d}x > 0$. (For those who have done *MT2502*, this can be easily deduced from the ε–δ definition of continuity.)]

2. If $A = [a_{ij}]$ is an $m \times n$ matrix over a field F, then its *transpose* A^{T} is the $n \times m$ matrix obtained by interchanging rows and columns: $A^{\mathsf{T}} = [a_{ji}]$ has (i,j)th entry equal to a_{ji}, which is the (j,i)th entry of A.

   (a) Let A be an $m \times n$ matrix and B be an $n \times r$ matrix over some field F. Show that

$$(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}.$$

   (b) Let A be an invertible matrix over a field F. Show that A^{T} is invertible and that

$$(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}.$$

   (c) Let A and B be $m \times n$ matrices over a field F and α be a scalar in F. Show that

$$(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}} \qquad \text{and} \qquad (\alpha A)^{\mathsf{T}} = \alpha\, A^{\mathsf{T}}.$$

3. Consider the vector space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ matrices with real entries. Verify that the following formula defines an inner product on $\mathrm{M}_n(\mathbb{R})$:

$$\langle A, B \rangle = \operatorname{tr}(AB^{\mathsf{T}}),$$

the trace of the product of A and the transpose of B.

[Recall that the *trace* of a square matrix is the sum of its diagonal entries.]

4. Let V be an inner product space having norm $\|\cdot\|$. Show that

$$\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$$

for all $v, w \in V$.

5. Let x_i and y_i be real numbers for $i = 1, 2, \dots, n$. Prove the following inequalities:

(a) $\displaystyle\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$;

(b) $\displaystyle\left(\sum_{i=1}^{n} x_i\right)^2 \leqslant n \sum_{i=1}^{n} x_i^2$.

[Hint: Use the Cauchy–Schwarz Inequality in an appropriate inner product space.]

6. Consider the vector space $\mathbb{R}^3$ with its usual inner product (given by the dot product). Apply the Gram–Schmidt process to the following bases to produce orthonormal bases for $\mathbb{R}^3$:

(a) $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$;

(b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}$.

7. Let $\mathcal{P}_2$ denote the inner product space of all polynomials over $\mathbb{C}$ of degree at most 2 with inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Apply the Gram–Schmidt process to the set $\{1, x, x^2\}$ to produce an orthonormal basis for $\mathcal{P}_2$.

8. Consider the space $\mathcal{P}_3$ of complex polynomials of degree at most 3 with inner product given by

$$\langle f, g \rangle = \int_{0}^{1} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Find an orthonormal basis for $\mathcal{P}_3$ by applying the Gram–Schmidt process to the standard basis of monomials $\{1, x, x^2, x^3\}$.

9. Consider the vector space $\mathbb{R}^3$. Let

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad U = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right).$$

Determine the orthogonal complement $U^{\perp}$.

Find the vector in U that is closest to $\boldsymbol{v}$ and hence determine the distance from $\boldsymbol{v}$ to U.

10. Let $\mathcal{P}_3$ be the space of complex polynomials of degree at most 3 and $U = \mathcal{P}_1$ be the subspace of polynomials of degree at most 1. Consider the following inner produce

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x.$$

Determine the orthogonal complement $U^\perp$ with respect to this inner product.

Find the polynomial in U that is closest to x^3 with respect to the norm determined by this inner product and hence determine the distance from x^3 to U.

11. Let V be a finite-dimensional inner product space.

    (a) Let w be a fixed vector in V. Show that the map $T\colon V \to V$ given by $T(v) = \langle v, w \rangle w$ is a linear map.
    Determine the image and the kernel of this map T.

    (b) Let U be a subspace of V and let $\{e_1, e_2, \dots, e_k\}$ be an orthonormal basis for U. Define $P\colon V \to V$ by

    $$P(v) = \sum_{i=1}^{k} \langle v, e_i \rangle e_i$$

    for $v \in V$. Show that P is the projection map onto U with respect to the orthogonal direct sum decomposition $V = U \oplus U^\perp$.
    [Hint: Show that $v - P(v)$ is orthogonal to e_j for each j.]