School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet V: Jordan normal form

1. Find all possible Jordan normal forms (discounting rearrangements of the Jordan blocks) for those linear transformations whose characteristic polynomial $c_T(x)$ and minimum polynomial $m_T(x)$ are as follows:

$$\begin{array}{ccc} & c_T(x) & m_T(x) \\ \text{(i)} & (x-2)^4(x-3)^2 & (x-2)^2(x-3)^2; \\ \text{(ii)} & (x-7)^5 & (x-7)^2; \\ \text{(iii)} & (x-2)^7 & (x-2)^3; \\ \text{(iv)} & (x-3)^4(x-5)^4 & (x-3)^2(x-5)^2. \end{array}$$

   [Hint: How can the dimension of the vector space be deduced from knowledge of the characteristic polynomial?]

2. Determine all possible Jordan normal forms for a linear transformation $T\colon V \to V$ whose characteristic polynomial is $c_T(x) = (x-2)^3(x-5)^2$.

3. Determine all possible Jordan normal forms for a 5×5 matrix whose minimum polynomial is $(x-3)^2$.

4. For each of the following matrices A of real numbers, determine a Jordan normal form of A:

$$\text{(i)} \begin{pmatrix} 3 & -4 & 0 \\ 0 & -1 & 0 \\ 0 & 6 & 2 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 4 & -2 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

$$\text{(iv)} \begin{pmatrix} 3 & 4 & 4 \\ 1 & 3 & 0 \\ -2 & -4 & -1 \end{pmatrix} \qquad \text{(v)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix} \qquad \text{(vi)} \begin{pmatrix} -2 & -3 & 0 \\ 3 & 4 & 0 \\ 6 & 6 & 1 \end{pmatrix}$$

   [Hint: These matrices appeared on Problem Sheet IV!]

5. For each matrix A appearing in Question 4, find an invertible matrix P such that $P^{-1}AP$ is in Jordan normal form.

6. Show that the matrix of real numbers

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

is not diagonalisable. Find a Jordan normal form J of A and an invertible matrix P such that $P^{-1}AP = J$.

7. For each of the following matrices A, determine a Jordan normal form of A and find an invertible matrix P such that $P^{-1}AP$ is in Jordan normal form.

(i) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -6 & -1 & 1 & 0 \\ -4 & 0 & -1 & 0 \\ 13 & 0 & 6 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -14 & 1 & 0 & 14 \\ -6 & 0 & 0 & 6 \\ 6 & -3 & -3 & -6 \\ -11 & 1 & 0 & 11 \end{pmatrix}$

(iii) $\begin{pmatrix} -2 & 1 & 0 & -9 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 7 \\ 1 & 0 & 0 & 4 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 & 0 & 2 & -4 \\ -2 & -2 & 3 & -4 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

(v) $\begin{pmatrix} -3 & 2 & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} -2 & 2 & 0 & -6 \\ -2 & 3 & 0 & -3 \\ -2 & 1 & 2 & -3 \\ 2 & -1 & 0 & 5 \end{pmatrix}$

8. (a) Let λ be a real number. Solve the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda x + y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \lambda y \end{aligned}$$

[This part does not involve any linear mathematics, but it is intended to prepare you for the second part.]

(b) Solve the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 9x + 4y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -25x - 11y \end{aligned}$$

[Hint: Find the Jordan normal form of the matrix $A = \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix}$.]