School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet IV: Diagonalisation

1. (a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation having matrix

$$A = \begin{pmatrix} -13 & -5 \\ 34 & 13 \end{pmatrix}$$

with respect to the standard basis. Calculate the characteristic polynomial of T and determine whether T is diagonalisable.

   (b) Let $S\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation having the above matrix A with respect to the standard basis. Is S diagonalisable?

2. Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space V. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of T.

   Show that T is diagonalisable if and only if V is the direct sum of the eigenspaces E_{λ_i} (for $i = 1, 2, \dots, k$).

   [Hint: Use the definition of diagonalisability (see Definition 4.6).]

3. For each matrix A below, let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation having matrix A with respect to the standard basis, that is,

$$\begin{aligned} T\colon \mathbb{R}^3 &\to \mathbb{R}^3 \\ \boldsymbol{v} &\mapsto A\boldsymbol{v}. \end{aligned}$$

   Calculate the characteristic polynomial of T, find the eigenvalues of T, find the algebraic and geometric multiplicities of each eigenvalue, and determine whether T is diagonalisable. If T is diagonalisable, find a matrix P such that $P^{-1}AP$ is diagonal.

$$\text{(i)} \begin{pmatrix} 3 & -4 & 0 \\ 0 & -1 & 0 \\ 0 & 6 & 2 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 4 & -2 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

$$\text{(iv)} \begin{pmatrix} 3 & 4 & 4 \\ 1 & 3 & 0 \\ -2 & -4 & -1 \end{pmatrix} \qquad \text{(v)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix} \qquad \text{(vi)} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\text{(vii)} \begin{pmatrix} -2 & -3 & 0 \\ 3 & 4 & 0 \\ 6 & 6 & 1 \end{pmatrix}$$

4. For each matrix in Question 3, determine the minimum polynomial of the corresponding transformation T.

5. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that A and B have different characteristic polynomials but have the same minimum polynomial.

6. Solve the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 5x \qquad - 6z \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 3x + y - 5z \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 3x \qquad - 4z \end{aligned}$$

7. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} -24 & -1 & -8 & -11 \\ 11 & 0 & 2 & 5 \\ 0 & 0 & 2 & 0 \\ 52 & 2 & 16 & 24 \end{pmatrix}.$$

Calculate the characteristic polynomial of T.

By considering the minimum polynomial, or otherwise, determine whether or not T is diagonalisable.

Determine the minimum polynomial of T.

8. Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space V over a field F. Let $f(x)$ be a polynomial over F, $\mathscr{B}$ be a basis for V and $A = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$. Show that

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(f(T)) = f(A).$$

[Hint: Use some of the questions on Problem Sheet II.]

9. Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space V over a field F. *Without using or depending on the Cayley–Hamilton Theorem*, show that every root of the minimum polynomial $m_T(x)$ is also a root of the characteristic polynomial $c_T(x)$.

[Hint: Let α be a root of $m_T(x)$ and factorize the polynomial as $m_T(x) = (x - \alpha)q(x)$. Now use the fact that $\deg q(x) < \deg m_T(x)$ to conclude something about $q(T)$.]

10. Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space V over a field F. Let $f(x)$ and $g(x)$ be *coprime polynomials* over F; that is, they are polynomials such that the only polynomials $h(x)$ that divide both $f(x)$ and $g(x)$ are constant polynomials (i.e., scalar multiples of 1).

    (a) Show that $\ker f(T)$ and $\ker g(T)$ are T*-invariant* (that is, show $T(v) \in \ker f(T)$ whenever $v \in \ker f(T)$ and similarly for $\ker g(T)$).

    (b) Show that $\ker f(T) \subseteq \ker f(T)g(T)$ and $\ker g(T) \subseteq \ker f(T)g(T)$.

    (c) Show that
    $$\ker f(T)g(T) = \ker f(T) \oplus \ker g(T).$$

    [Hint for (c): As a Euclidean domain, the set $F[x]$ of polynomials over F shares the following property with the integers. If $f(x)$ and $g(x)$ are coprime polynomials, then there exist polynomials $u(x)$ and $v(x)$ such that
    $$1 = u(x)f(x) + v(x)g(x).$$
    You may assume this fact. It will be established in *MT3505 Rings and Fields*.]