School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet III: Direct sums

1. Consider the vector space $\mathbb{R}^4$. Let

$$U = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right) \qquad \text{and} \qquad W = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right).$$

   Show that $\mathbb{R}^4 = U \oplus W$.

   Let $P \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the projection map onto U. Calculate

$$P \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. Let $V = U \oplus W$ be a finite-dimensional vector space which is the direct sum of two subspaces U and W. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for U and W, respectively.

   What is the matrix of the projection map $P \colon V \to V$ onto U with respect to the basis $\mathscr{B}_1 \cup \mathscr{B}_2$ for V?

3. Let V be a finite-dimensional vector space with basis $\mathscr{B}$. If $\mathscr{A}$ is a subset of $\mathscr{B}$ and $\mathscr{B} \setminus \mathscr{A}$ denotes the complement of $\mathscr{A}$ in $\mathscr{B}$, show that

$$V = \operatorname{Span}(\mathscr{A}) \oplus \operatorname{Span}(\mathscr{B} \setminus \mathscr{A}).$$

4. Let U be the following subspace of $\mathbb{R}^3$:

$$U = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

   Find a subspace W of $\mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

   Calculate the matrices of the corresponding projection maps $P, Q \colon \mathbb{R}^3 \to \mathbb{R}^3$ onto U and W, respectively with respect to the standard basis for $\mathbb{R}^3$.

   Check explicitly that $\ker P = W$ and $\operatorname{im} P = U$.

5. Let V be a finite-dimensional vector space and let U be a subspace of V. Show that there exists a subspace W such that $V = U \oplus W$.

   [Hint: Extend a basis for U to a basis for V.]

6. A linear transformation $T: V \to V$ is called *idempotent* if $T^2 = T$; that is, if

$$T(T(v)) = T(v) \qquad \text{for all } v \in V.$$

Let $T: V \to V$ be an idempotent linear transformation.

(a) Show that $v - T(v)$ lies in the kernel of T for all $v \in V$.

(b) Show that $V = \ker T + \operatorname{im} T$. [Hint: Use (a).]

(c) Show that $\ker T \cap \operatorname{im} T = \{\mathbf{0}\}$.

(d) Parts (b) and (c) tell us that $V = \ker T \oplus \operatorname{im} T$. Let P be the associated projection onto the direct summand $U = \operatorname{im} T$. Show that $P = T$.

(e) Deduce that idempotent linear transformations and projection maps are precisely the same things.

7. Let $T: V \to V$ be an idempotent linear transformation. Prove directly from the definition that $I - T$ is also an idempotent transformation.

[Here $I: V \to V$ is the identity map. By "directly from the definition", I mean that you should not use the fact that idempotent transformations and projection maps are precisely the same objects.]

8. Let V be a vector space and let $U_1, U_2, \dots, U_k$ be subspaces of V. Show that every vector in V can be uniquely expressed in the form $u_1 + u_2 + \dots + u_k$ with $u_i \in U_i$ for each i if and only if the following two conditions hold:

(a) $V = U_1 + U_2 + \dots + U_k$;

(b) $U_i \cap (U_1 + \dots + U_{i-1} + U_{i+1} + \dots + U_k) = \{\mathbf{0}\}$ for each i.