School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet II: Linear transformations

1. Define a function $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 3y - z \\ x + 2y - 2z \\ -x + 4z \end{pmatrix}.$$

   (a) Show that T is a linear transformation.

   (b) Determine the kernel of T and find a basis for $\ker T$. [Hint: Solving $T(\boldsymbol{v}) = \mathbf{0}$ will correspond to solving a set of simultaneous linear equations.]

   (c) Show that

$$\operatorname{im} T = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} \right).$$

   Hence find a basis for $\operatorname{im} T$.

   (d) Verify the Rank-Nullity Theorem holds for this specific example.

2. Define a linear transformation $T \colon \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -3 \\ 0 \\ 9 \end{pmatrix},$$

$$T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_4) = \boldsymbol{y}_4 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

   where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the standard basis for $\mathbb{R}^4$.

   (a) Find a subset of $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4\}$ that is a basis for the image of T.

   (b) Find a basis for the kernel of T.

   (c) Hence determine the rank and nullity of T.

3. Let V and W be finite-dimensional vector spaces over a field F. An *isomorphism* between V and W is a linear transformation $T\colon V \to W$ which is invertible (that is, there is an inverse, i.e., a linear transformation $S\colon W \to V$ such that $ST = I$ and $TS = I$ are the identity maps). We say that V and W are *isomorphic*, written $V \cong W$, if there exists an isomorphism $V \to W$.

   (a) Show that a linear transformation $T\colon V \to W$ is injective (that is, $T(u) = T(v)$ implies $u = v$) if and only if $\ker T = \{\mathbf{0}\}$.

   (b) Show that a linear transformation $T\colon V \to W$ is an isomorphism if and only if it is bijective (that is, both injective and surjective (which means $\operatorname{im} T = W$)).

   (c) If $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ is a basis for W, show that the unique linear transformation $T\colon V \to W$ given by $T(v_i) = w_i$, for $i = 1$, 2, $\dots$, n, is an isomorphism.
   Deduce that $V \cong W$ if and only if $\dim V = \dim W$.

4. Let V be a finite-dimensional vector space and $T\colon V \to V$ be a linear map. Show that the following conditions on T are equivalent:

   (a) $\ker T = \{0\}$;

   (b) $\operatorname{im} T = V$;

   (c) T is invertible.

   [Hint: Use the Rank-Nullity Theorem.]

5. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping whose matrix with respect to the standard basis for $\mathbb{R}^2$ is

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix};$$

   that is, $T(\boldsymbol{v}) = A\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^2$.

   (a) Show that

$$\mathscr{B} = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \end{pmatrix} \right\}$$

   is a basis for $\mathbb{R}^2$.

   (b) Calculate the matrix $\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T)$ of T with respect to the basis $\mathscr{B}$.

6. Define the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 2z \\ -3x + 4y - 2z \\ -2y \end{pmatrix}.$$

   (a) Find the matrix of T with respect to the standard basis for $\mathbb{R}^3$.

   (b) Show that

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

   is a basis for $\mathbb{R}^3$.

   (c) Find the matrix of T with respect to the basis $\mathscr{B}$.

7. Let U, V and W be vector spaces over the field F. If $T: U \to V$ and $S: V \to W$ are linear transformations, show that the composition

$$\begin{aligned} ST: U &\to W \\ u &\mapsto S(Tu) \end{aligned}$$

is also a linear transformation.

Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be bases for U, V and W, respectively. Show that

$$\mathrm{Mat}_{\mathscr{A},\mathscr{C}}(ST) = \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(S) \cdot \mathrm{Mat}_{\mathscr{A},\mathscr{B}}(T).$$

8. The purpose of this question is to establish the change of basis formula (Theorem 2.12 in the lecture notes) in a more elegant manner using the formula established in the previous question.

Let V be a finite-dimensional vector space and let $\mathscr{B}$ and $\mathscr{C}$ be bases for V. Let $T: V \to V$ be a linear map.

(a) Show that the change of basis matrix P obtained by writing each vector in $\mathscr{C}$ in terms of the basis $\mathscr{B}$ is equal to the matrix $\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(I)$ of the identity map $I: V \to V$ with respect to these bases.

(b) Deduce, using the formula in Question 7, that $\mathrm{Mat}_{\mathscr{B},\mathscr{C}}(I) = \mathrm{Mat}_{\mathscr{C},\mathscr{B}}(I)^{-1}$.

(c) Use the formula in Question 7 to show that

$$\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(I) \cdot \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) \cdot \mathrm{Mat}_{\mathscr{C},\mathscr{B}}(I).$$

Deduce that

$$\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = P^{-1} \cdot \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) \cdot P.$$

9. (a) Let F be a field and A be an $m \times n$ matrix. Show that A defines a linear transformation $F^n \to F^m$ by

$$\boldsymbol{v} \mapsto A\boldsymbol{v} \qquad \text{for } \boldsymbol{v} \in F^n.$$

Show that the matrix of this linear transformation with respect to the standard bases of F^n and F^m is given by $\mathrm{Mat}(A) = A$.

(b) Now consider any linear transformation $T: F^n \to F^m$. Show that

$$T(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in F^n,$$

where A is the matrix of T with respect to the standard bases for F^n and F^m. [Thus, every linear transformation from F^n to F^m is given by matrix multiplication.]

10. Let V and W be vector spaces over the field F. Recall that, if S and T are linear transformations and $\alpha \in F$ then we have defined linear maps $S + T$ and αT by

$$(S + T)(v) = S(v) + T(v) \qquad \text{and} \qquad (\alpha T)(v) = \alpha \cdot T(v)$$

for $v \in V$. (See Definition 2.14 and Lemma 2.15.)

Show that $\mathcal{L}(V, W)$, the set of all linear maps $V \to W$, is itself a vector space over F with the above addition and scalar multiplication.

11. Let V and W be finite-dimensional vector spaces over the field F having bases $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_m\}$, respectively.

    (a) Let $S, T\colon V \to W$ be linear maps and $\alpha \in F$. Show that

    $$\mathrm{Mat}_{\mathscr{B},\mathscr{C}}(S+T) = \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(S) + \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T)$$

    and

    $$\mathrm{Mat}_{\mathscr{B},\mathscr{C}}(\alpha T) = \alpha \cdot \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T).$$

    (b) Show that $T \mapsto \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T)$ defines an isomorphism from $\mathcal{L}(V, W)$ to the vector space $\mathrm{M}_{m\times n}(F)$ of $m \times n$ matrices over F.

    (c) Deduce that $\dim \mathcal{L}(V, W) = mn$.

12. Let V be a finite-dimensional vector space over a field F with basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$. Define a linear map $f_i\colon V \to F$ by

    $$f_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

    Show that $\{f_1, f_2, \dots, f_n\}$ is a basis for the dual space V^*.

13. A linear transformation $T\colon V \to V$ is said to be *nilpotent* of *index* k if T^k is the zero map but T^{k-1} is not.

    Suppose that V is a vector space of dimension n and the linear transformation $T\colon V \to V$ is nilpotent of index n. Choose a vector v such that $T^{n-1}(v) \neq \mathbf{0}$. Show that

    $$\mathscr{B} = \{v, T(v), T^2(v), \dots, T^{n-1}(v)\}$$

    is a basis for V. [Hint: Show that it is linearly independent. Consider an expression of the form $\sum_{i=0}^{n-1} \alpha_i T^i(v) = \mathbf{0}$ and apply a suitable power of T.]

    Write down the matrix of T with respect to $\mathscr{B}$.

    Now consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect to the standard basis is

    $$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

    Show that T is nilpotent of index 3. Find a basis with respect to which T has matrix

    $$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$