School of Mathematics and Statistics

MT3501 Linear Mathematics 2

Problem Sheet I: Vector spaces

1. Let $\mathcal{F}_{\mathbb{R}}$ denote the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$. Show that $\mathcal{F}_{\mathbb{R}}$ is a vector space over $\mathbb{R}$ under the operations given by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for } x \in \mathbb{R}$$

and, for $\alpha \in \mathbb{R}$,

$$(\alpha f)(x) = \alpha \cdot f(x) \qquad \text{for } x \in \mathbb{R}.$$

2. Let F be a field and let $\mathrm{M}_n(F)$ denote the set of all $n \times n$ matrices over F.

   (a) Show that $\mathrm{M}_n(F)$ is a vector space over F with respect to the usual addition of matrices and where scalar multiplication is given by

$$\alpha \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \cdots & \alpha x_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha x_{n1} & \alpha x_{n2} & \cdots & \alpha x_{nn} \end{pmatrix}$$

   for $\alpha \in F$ and $[x_{ij}] \in \mathrm{M}_n(F)$. What is the zero vector in this vector space?

   (b) Let W consist of the set of *symmetric* matrices in $\mathrm{M}_n(F)$. (A matrix $[x_{ij}]$ is symmetric if $x_{ij} = x_{ji}$ for all i and j.) Show that W is a subspace of $\mathrm{M}_n(F)$.

   (c) Fix a matrix A in $\mathrm{M}_n(F)$ and let

$$V = \{\, X \in \mathrm{M}_n(F) \mid AX = XA \,\}.$$

   Show that V is a subspace of $\mathrm{M}_n(F)$.

   (d) Let Z be the set of matrices in $\mathrm{M}_n(F)$ of zero determinant. If $n > 1$, show that Z is not a subspace of $\mathrm{M}_n(F)$.

   (e) Take $F = \mathbb{R}$ and let I be the set of matrices B in $\mathrm{M}_n(\mathbb{R})$ satisfying $B^2 = B$. Show that I is not a subspace of $\mathrm{M}_n(\mathbb{R})$.

3. Consider the vector space $\mathbb{R}[x]$ of polynomials over the real numbers. Determine whether the following subsets are subspaces of $\mathbb{R}[x]$:

   (a) $\{\, f(x) \mid f(1) = 0 \,\}$,

   (b) $\{\, f(x) \mid \text{the constant term of } f(x) \text{ is } 0 \,\}$,

   (c) $\{\, f(x) \mid f(x) \text{ is a polynomial of degree precisely } 3 \,\}$,

   (d) $\{\, f(x) \mid f(x) \text{ is a polynomial of degree at most } 3 \,\}$,

   (e) $\{\, f(x) \mid f(x) \text{ is a polynomial of even degree} \,\}$.

4. Let W_1 and W_2 be the following subspaces of the space $\mathbb{R}[x]$ of polynomials over the real numbers:

$$W_1 = \{\, f(x) \mid f(1) = 0 \,\} \qquad \text{and} \qquad W_2 = \{\, f(x) \mid f(2) = 0 \,\}.$$

Give a simple description of $W_1 \cap W_2$. Show that every element of $\mathbb{R}[x]$ can be written in the form

$$f(x) = g_1(x) + g_2(x)$$

where $g_1(x) \in W_1$ and $g_2(x) \in W_2$. Is this decomposition unique?

5. Show that

$$W = \left\{ \begin{pmatrix} x+y-z \\ y-x+z \\ 2x \end{pmatrix} \;\middle|\; x, y, z \in R \right\}$$

is a subspace of $\mathbb{R}^3$. Show that

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a spanning set for W. Is $\mathscr{A}$ linearly independent? Determine the dimension of W and hence whether or not $W = \mathbb{R}^3$.

6. Let V be a vector space and let W_1 and W_2 be finite-dimensional subspaces of V. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for W_1 and W_2, respectively, show that $\mathscr{B}_1 \cup \mathscr{B}_2$ is a spanning set for $W_1 + W_2$.

Is $\mathscr{B}_1 \cup \mathscr{B}_2$ necesssarily a basis for $W_1 + W_2$? Provide a counterexample if not.

7. Let V be a vector space and v and w be vectors in V. Show that $\operatorname{Span}(v) = \operatorname{Span}(w)$ if and only if $w = \alpha v$ for some non-zero scalar α.

8. Let

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 4 \\ 8 \\ -4 \\ -3 \end{pmatrix}, \qquad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix},$$

$$\boldsymbol{v}_4 = \begin{pmatrix} 1 \\ 10 \\ -6 \\ -2 \end{pmatrix}, \qquad \boldsymbol{v}_5 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_6 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 4 \end{pmatrix}$$

be six vectors in the vector space $\mathbb{R}^4$. Let $U = \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$ and $W = \operatorname{Span}(\boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6)$. Determine $\dim U$, $\dim W$ and $\dim(U + W)$.

[Hint: Determine whether $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly independent. If not, pass to a subset which also spans U. Similarly, for W and $U + W$, making use of the previous question to provide spanning set for the sum.]

9. Consider the following set $\mathscr{A}$ of vectors in $\mathbb{R}^3$ or $\mathbb{R}^4$ (respectively). Find a subset $\mathscr{B}$ of $\mathscr{A}$ that is a basis for the subspace $U = \operatorname{Span}(\mathscr{A})$.

(a) $\mathscr{A} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \right\}$

(b) $\mathscr{A} = \left\{ \begin{pmatrix} 0 \\ 2 \\ -3 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ -1 \\ -15 \end{pmatrix} \right\}$

10. Let V be the subspace of the space $\mathbb{R}[x]$ of real polynomials spanned by the following polynomials:

$$f_1(x) = x^3 + 2x^2 + 1, \qquad f_2(x) = x^2 + 3x + 4, \qquad f_3(x) = 2x^3 - 12x - 2.$$

Find a subset $\mathscr{A}$ of $\{f_1(x), f_2(x), f_3(x)\}$ which is a basis for V and hence determine the dimension of V.

Find a basis $\mathscr{B}$ for the subspace $\mathbb{R}[x]$ consisting of polynomials of degree at most 4 such that $\mathscr{A} \subseteq \mathscr{B}$. [That is, extend $\mathscr{A}$ to a basis $\mathscr{B}$ for the space of polynomials of degree at most 4.]

11. Let V be a finite-dimensional vector space and W be a subspace of V.

(a) Show that W is finite-dimensional and that $\dim W \leqslant \dim V$.

(b) Show that $V = W$ if and only if $\dim V = \dim W$.

12. Let V be a finite-dimensional vector space with subspaces U and W. Let

$$\mathscr{A} = \{v_1, v_2, \dots, v_k\}$$

be a basis for $U \cap W$. Extend $\mathscr{A}$ to a basis

$$\mathscr{B}_1 = \{v_1, v_2, \dots, v_k, u_{k+1}, \dots, u_m\}$$

for U and extend $\mathscr{A}$ to a basis

$$\mathscr{B}_2 = \{v_1, v_2, \dots, v_k, w_{k+1}, \dots, w_n\}$$

for W.

Show that the set

$$\mathscr{B}_1 \cup \mathscr{B}_2 = \{v_1, v_2, \dots, v_k, u_{k+1}, \dots, u_m, w_{k+1}, \dots, w_n\}$$

is a basis for $U + W$.

Deduce that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$