MT3501

Linear Mathematics 2

MRQ

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University of St Andrews

Contents

Introduction

The module description for *MT3501* says that

> "It aims to show the importance of linearity in many areas of mathematics ranging from linear algebra through geometric applications to linear operators and special functions."

A student in their Honours programme will already have begun to develop some experience of the sort of objects that occur in mathematics and to develop some facility in working with them. Functions are extremely important in mathematics and to date probably the most studied object in a student's programme. In this module, we shall see how functions that are linear are far more tractable to study. We can obtain information about linear mappings that we would not attempt to find for arbitrary functions. Moreover, linearity occurs naturally throughout mathematics and for this reason understanding linear mappings is vital for the study of pure, applied and statistical mathematics. The obvious example is differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}$$

(when c is a constant).

We start by discussing vector spaces, since these are the correct and natural place within which to frame linearity. They may initially look like an abstract algebraic concept, but they do illustrate the two main themes of this course:

(i) Vector spaces are, in some sense, among the easiest algebraic structure to study; we can establish facts that we would hope to be true more easily than in other algebraic systems. (For example, in groups many such results do not even hold, though arguably this makes for interesting mathematics.)

(ii) An abstract setting is good to work within since we can establish facts in general and then apply them to many settings. Remember (for example, from *MT2501*) that linearity occurs throughout mathematics: rather than proving something just for Euclidean vectors, we can obtain results that cover vectors, complex numbers, matrices, polynomials, differentiable functions, solutions to differential equations, etc.

Furthermore, the methods of linear mathematics have application in many diverse areas. We shall attempt to illustrate just a sample of these applications as we develop the material in the course.

Overview of course content

- Vector spaces: subspaces, spanning sets, linear independent sets, bases

- Linear transformations: rank, nullity, general form of a linear transformation, matrix of a linear transformation, change of basis
- Direct sums: projection maps
- Diagonalisation of linear transformations: eigenvectors and eigenvalues, characteristic polynomial, minimum polynomial, characterisation of diagonalisable transformations
- Jordan normal form: method to determine the Jordan normal form
- Inner product spaces: orthogonality, associated inequalities, orthonormal bases, Gram–Schmidt process, applications
- Examples of infinite-dimensional inner product spaces

Additional examples

The lecture notes (as available via Moodle) contain a number of examples that will not be covered on the board during the actual lectures. This is an attempt to provide more examples, while not decreasing the amount of material covered. These omitted examples are numbered in the form "**1A**" (etc.) in the body of these notes.

Recommended texts

- Sheldon Axler, *Linear Algebra Done Right, 3rd Edition*, Undergraduate Texts in Mathematics (Springer 2015)
- T. S. Blyth & E. F. Robertson, *Basic Linear Algebra, Second Edition*, Springer Undergraduate Mathematics Series (Springer-Verlag 2002)
- T. S. Blyth & E. F. Robertson, *Further Linear Algebra*, Springer Undergraduate Mathematics Series (Springer-Verlag 2002)
- R. Kaye & R. Wilson, *Linear Algebra*, Oxford Science Publications (OUP 1998)

Chapter 1

Vector spaces

The purpose of this chapter is to review material from *MT2501* (*Linear Mathematics*) concerning vector spaces and linear independence. No new material is covered, but the purpose is to set up the notation and remind you what was done in that prerequisite module. In the lectures this material will be reviewed quite speedily, most proofs will be omitted and those that do get covered are to review how this sort of mathematics is performed. Worked examples appear in the lecture notes but are likely to be omitted in the actual lectures. (These examples are all of the same style as covered in *MT2501* and are included in the notes for revision purposes.)

Definition and examples of vector spaces

Being "linear" will boil down to "preserving addition" and "preserving scalar multiplication." By the term *scalar*, we mean an element of a field, so our first step is to make the following definition:

Definition 1.1 A *field* is a set F together with two binary operations

$$\begin{aligned} F \times F &\to F \\ (\alpha, \beta) &\mapsto \alpha + \beta \end{aligned} \qquad\qquad \begin{aligned} F \times F &\to F \\ (\alpha, \beta) &\mapsto \alpha\beta \end{aligned}$$

called *addition* and *multiplication*, respectively, such that

(A1) $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in F$;

(A2) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all $\alpha, \beta, \gamma \in F$;

(A3) there exists an element 0 in F such that $\alpha + 0 = \alpha$ for all $\alpha \in F$;

(A4) for each $\alpha \in F$, there exists an element $-\alpha$ in F such that $\alpha + (-\alpha) = 0$;

(M1) $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in F$;

(M2) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in F$;

(D) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all $\alpha, \beta, \gamma \in F$;

(M3) there exists an element 1 in F such that $1 \neq 0$ and $1\alpha = \alpha$ for all $\alpha \in F$;

(M4) for each $\alpha \in F$ with $\alpha \neq 0$, there exists an element α^{-1} (or $1/\alpha$) in F such that $\alpha\alpha^{-1} = 1$.

The labels on the above conditions are chosen to match those used in *MT2505 Abstract Algebra* and those that will be used in *MT3505 Rings and Fields*. Although a full set of axioms have been provided, we are not going to examine them in detail nor spend developing the theory of fields. (Those who are interested should in pursuing this further should enrol in *MT3505*.) Instead, one should interpret the above definition as saying that in a field one may add, subtract, multiply and divide (by *non-zero* scalars) and that all the familiar rules of arithmetic hold. This is reflected in the list of examples relevant to the module:

Example 1.2 The following are examples of fields:

(i) the field of rational numbers $\mathbb{Q} = \{\, m/n \mid m, n \in \mathbb{Z},\ n \neq 0 \,\}$;

(ii) the field $\mathbb{R}$ of real numbers;

(iii) the field of complex numbers $\mathbb{C} = \{\, x + iy \mid x, y \in \mathbb{R} \,\}$

with all three possessing the usual addition and multiplication;

(iv) $\mathbb{F}_p = \{0, 1, \dots, p-1\}$, where p is a prime number, with addition and multiplication being performed modulo p.

The last example was introduced in *MT2505 Abstract Algebra*. For the purposes of this module and for many applications of linear algebra in applied mathematics and the physical sciences, the examples $\mathbb{R}$ and $\mathbb{C}$ are the most important and it is safe to think of them as the typical examples of a field throughout. For those of a pure mathematical bent, however, it is worth noting that some parts of what is done in this module will work over an arbitrary field.

Definition 1.3 Let F be a field. A *vector space* over F is a set V together with the following operations

$$\begin{aligned} V \times V &\to V \\ (u, v) &\mapsto u + v \end{aligned} \qquad\qquad \begin{aligned} F \times V &\to V \\ (\alpha, v) &\mapsto \alpha v, \end{aligned}$$

called *addition* and *scalar multiplication*, respectively, such that

(V1) $u + v = v + u$ for all $u, v \in V$;

(V2) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$;

(V3) there exists a vector $\mathbf{0}$ in V such that $v + \mathbf{0} = v$ for all $v \in V$;

(V4) for each $v \in V$, there exists a vector $-v$ in V such that $v + (-v) = \mathbf{0}$;

(V5) $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in V$ and $\alpha \in F$;

(V6) $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in V$ and $\alpha, \beta \in F$;

(V7) $(\alpha\beta)v = \alpha(\beta v)$ for all $v \in V$ and $\alpha, \beta \in F$;

(V8) $1v = v$ for all $v \in V$.

Comments:

(i) A vector space then consists of a collection of *vectors* which we are permitted to add and which we may multiply by *scalars* from our base field. The conditions assumed (that have been labelled to coincide with that used in *MT2501 Linear Mathematics*) ensure that these operations behave in a familiar way and that we can manipulate expressions involving vectors and scalars without issue.

(ii) One aspect which requires some care is that the field contains the number 0 while the vector space contains the zero vector $\mathbf{0}$. The latter will be denoted by boldface in the lecture notes and on the slides. On the board boldface is unavailable, so although the difference is usually clear from the context (we can multiply vectors by scalars, but are not permitted to multiply vectors, and we can add two vectors but cannot add a vector to a scalar) we shall use $\underset{\sim}{0}$ to denote the zero vector on that medium.

(iii) We shall use the term *real vector space* to refer to a vector space over the field $\mathbb{R}$ and *complex vector space* to refer to one over the field $\mathbb{C}$. Almost all examples in this module will be either real or complex vector spaces.

(iv) We shall sometimes refer simply to a vector space V without specifying the base field F. Nevertheless, there is always such a field F in the background and we will then use the term *scalar* to refer to the elements of this field, even though we may have failed to name what this field is.

To illustrate why these are important objects, we shall give a number of examples, many of which should be familiar (not least from *MT2501*).

Example 1.4 The following are examples of vector spaces. The verification in each case that it is a vector space is straightforward and most were covered in *MT2501*. The purpose of this list is to remind you of these standard examples of vector spaces for future use.

(i) Let n be a positive integer and let F^n denote the set of column vectors of length n with entries from the field F:

$$F^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_1, x_2, \dots, x_n \in F \right\}.$$

Addition and scalar multiplication in this vector space are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \qquad \text{and} \qquad \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}.$$

We could also consider the set of row vectors of length n as a vector space over F. This is sometimes also denoted F^n and has the advantage of being more easily written on the page! However, column vectors turn out to be slightly more natural than row vectors when we consider the matrix of a linear transformation later in the next chapter.

As a further comment about distinguishing between scalars and vectors, we shall follow the usual convention of using boldface (or writing something such as $\underset{\sim}{v}$ on the board) to denote column vectors in the vector space F^n. We shall, however, usually not use boldface letters when referring to vectors in an abstract vector space (since in an actual example, they could become genuine column vectors, but also possibly matrices, polynomials, functions, etc.).

(ii) The set of complex numbers $\mathbb{C}$ can be viewed as a vector space over $\mathbb{R}$. Addition is the usual addition of complex numbers:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2);$$

while scalar multiplication is given by

$$\alpha(x + iy) = (\alpha x) + i(\alpha y) \qquad (\text{for } \alpha \in \mathbb{R}).$$

The zero vector is the element $0 = 0 + i0 \in \mathbb{C}$. [Note that this is an example where the "vectors" are not column vectors, but are complex numbers. Accordingly, we choose in this example *not* to write the "vectors" as $\boldsymbol{v}$ or $\underset{\sim}{v}$ since we will stick to the usual tradition of writing $z = x + iy$ for a complex number.]

(iii) A *polynomial* over the field F is an expression of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m,$$

for some $m \geqslant 0$, where $a_0, a_1, \ldots, a_m \in F$ and where we ignore terms with 0 as the coefficient. The set of all polynomials over F is usually denoted by $F[x]$. If necessary we can "pad" such an expression for a polynomial using 0 as the coefficient for the extra terms to increase its length. Thus to add $f(x)$ above to another polynomial $g(x)$, we may assume they are represented by expressions of the same length, say

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m.$$

Then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_m + b_m)x^m.$$

Scalar multiplication is given by

$$\alpha f(x) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_m)x^m$$

for $f(x)$ as above and $\alpha \in F$.

(iv) The final example is related to the example of polynomials, since they are special types of functions. Let $\mathcal{F}_{\mathbb{R}}$ denote the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$. Define the addition of two functions f and g by

$$(f + g)(x) = f(x) + g(x) \qquad (\text{for } x \in \mathbb{R})$$

and scalar multiplication of f by $\alpha \in \mathbb{R}$ by

$$(\alpha f)(x) = \alpha \cdot f(x) \qquad (\text{for } x \in \mathbb{R}).$$

Then $\mathcal{F}_{\mathbb{R}}$ is a real vector space with these operations.

These examples illustrate that vector spaces occur in numerous situations (functions covering a large class of mathematical objects for a start) and so the study of linear algebra is of considerable importance.

Basic properties of vector spaces

The following basic properties demonstrate that the conditions defining a vector space ensure that its operations behave in familiar ways.

Proposition 1.5 *Let V be a vector space over a field F. Let $v \in V$ and $\alpha \in F$. Then*

(i) $\alpha \mathbf{0} = \mathbf{0}$;

(ii) $0v = \mathbf{0}$;

(iii) *if* $\alpha v = \mathbf{0}$, *then either* $\alpha = 0$ *or* $v = \mathbf{0}$;

(iv) $(-\alpha)v = -\alpha v = \alpha(-v)$.

PROOF (OMITTED IN LECTURES): This proof was covered in *MT2501*.

(i) Use Condition **(V5)** of Definition 1.3 to give

$$\alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0};$$

that is,

$$\alpha\mathbf{0} = \alpha\mathbf{0} + \alpha\mathbf{0}.$$

Now add $-\alpha\mathbf{0}$ to both sides to yield

$$\mathbf{0} = \alpha\mathbf{0}.$$

(ii) Use Condition **(V6)** of Definition 1.3 to give

$$(0 + 0)v = 0v + 0v;$$

that is,

$$0v = 0v + 0v.$$

Now add $-0v$ to deduce $\mathbf{0} = 0v$.

(iii) Suppose $\alpha v = \mathbf{0}$, but that $\alpha \neq 0$. Then F contains the scalar α^{-1} and multiplying by this gives

$$\alpha^{-1}(\alpha v) = \alpha^{-1}\mathbf{0} = \mathbf{0} \qquad \text{(using part (i))}.$$

Therefore

$$1v = (\alpha^{-1} \cdot \alpha)v = \mathbf{0}.$$

Condition **(V8)** of Definition 1.3 then shows $v = \mathbf{0}$. Hence if $\alpha v = \mathbf{0}$, either $\alpha = 0$ or $v = \mathbf{0}$.

(iv) Observe

$$\alpha v + (-\alpha)v = (\alpha + (-\alpha))v = 0v = \mathbf{0},$$

so $(-\alpha)v$ is the vector which when added to αv yields $\mathbf{0}$; that is, $(-\alpha)v = -\alpha v$.

Similarly,

$$\alpha v + \alpha(-v) = \alpha(v + (-v)) = \alpha\mathbf{0} = \mathbf{0}$$

and we deduce that $\alpha(-v)$ must be the vector $-\alpha v$. □

Subspaces

The concept of a subspace was introduced in *MT2501*. These will appear throughout the new material in this course and so we recall the definition and basic properties.

Definition 1.6 Let V be a vector space over a field F. A *subspace* W of V is a non-empty subset of V such that

(i) if $u, v \in W$, then $u + v \in W$, and

(ii) if $v \in W$ and $\alpha \in F$, then $\alpha v \in W$.

Thus, in words, a subspace W is a non-empty subset of the vector space V such that W is closed under vector addition and scalar multiplication by any scalar from the field F. The following basic properties hold:

Lemma 1.7 *Let V be a vector space and let W be a subspace of V. Then*

(i) $\mathbf{0} \in W$*;*

(ii) *if $v \in W$, then $-v \in W$.*

Proof (Omitted in lectures): These observations were covered in *MT2501*.

(i) Since W is non-empty, there exists at least one vector $u \in W$. Now W is closed under scalar multiplication, so $0u \in W$; that is, $\mathbf{0} \in W$ (by Proposition 1.5(ii)).

(ii) Let v be any vector in W. Then W contains

$$(-1)v = -1v = -v. \qquad \square$$

Consequence: If W is a subspace of V (over a field F), then W is also a vector space of F: addition and scalar multiplication are defined on W according to Definition 1.6. The axioms are inherited from the fact that they hold in the original space V. For example, the original zero vector $\mathbf{0}$ from V also is in W and satisfies the property for it to be the zero vector in the space W also.

Example 1.8 Many examples of subspaces were presented in *MT2501*. We list a few here:

(i) Let $V = \mathbb{R}^3$, the real vector space of column vectors of length 3. Consider

$$W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3;$$

so W consists of all vectors with zero in the last entry. We check

$$\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{pmatrix} \in W$$

and

$$\alpha \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ 0 \end{pmatrix} \in W \qquad (\text{for } \alpha \in \mathbb{R}).$$

Thus W is closed under sums and scalar multiplication; that is, W is a subspace of $\mathbb{R}^3$.

(ii) Let $\mathcal{F}_{\mathbb{R}}$ be the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$, which forms a real vector space under

$$(f+g)(x) = f(x) + g(x); \qquad (\alpha f)(x) = \alpha \cdot f(x).$$

Let $\mathcal{P}$ denote the set of polynomial functions; i.e., each $f \in \mathcal{P}$ has the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

for some $m \geqslant 0$ and $a_0, a_1, \dots, a_m \in \mathbb{R}$. Then $\mathcal{P} \subseteq \mathcal{F}_{\mathbb{R}}$ and, since the sum of two polynomials is a polynomial and a scalar multiple of a polynomial is a polynomial, $\mathcal{P}$ is a subspace of $\mathcal{F}_{\mathbb{R}}$.

We shall meet a generic way of constructing subspaces in a short while. The following specifies basic ways of manipulating subspaces.

Definition 1.9 Let V be a vector space and let U and W be subspaces of V.

(i) The *intersection* of U and W is

$$U \cap W = \{\, v \mid v \in U \text{ and } v \in W \,\}.$$

(ii) The *sum* of U and W is

$$U + W = \{\, u + w \mid u \in U, w \in W \,\}.$$

Since V is a vector space, addition of a vector $u \in U \subseteq V$ and $w \in W \subseteq V$ makes sense. Thus the sum $U + W$ is a sensible collection of vectors in V.

Proposition 1.10 *Let V be a vector space and let U and W be subspaces of V. Then*

(i) *$U \cap W$ is a subspace of V;*

(ii) *$U + W$ is a subspace of V.*

Proof: Both assertions were verified in *MT2501*. The proof of (i) will be omitted in lectures, but part (ii) will be covered as it is particularly relevant to concepts introduced in Chapter 3 and later.

(i) (Omitted in lectures) First note that Lemma 1.7(i) tells us that $\mathbf{0}$ lies in both U and W. Hence $\mathbf{0} \in U \cap W$, so this intersection is non-empty. Let $u, v \in U \cap W$ and α be a scalar from the base field. Then U is a subspace containing u and v, so $u + v \in U$ and $\alpha v \in U$. Equally, $u, v \in W$ so we deduce $u + v \in W$ and $\alpha v \in W$. Hence $u + v \in U \cap W$ and $\alpha v \in U \cap W$. This shows $U \cap W$ is a subspace of V.

(ii) Using the fact that $\mathbf{0}$ lies in U and W, we see $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$. Hence $U + W$ is non-empty. Now let $v_1, v_2 \in U + W$, say $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ where $u_1, u_2 \in U$ and $w_1, w_2 \in W$. Then

$$v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$$

and if α is a scalar then

$$\alpha v_1 = \alpha(u_1 + w_1) = (\alpha u_1) + (\alpha w_1) \in U + W.$$

Hence $U + W$ is a subspace of V. □

A straightforward induction argument then establishes:

Corollary 1.11 *Let V be a vector space and let U_1, U_2, …, U_k be subspaces of V. Then*

$$U_1 + U_2 + \cdots + U_k = \{\, u_1 + u_2 + \cdots + u_k \mid u_i \in U_i \text{ for each } i \,\}$$

is a subspace of V. □

Spanning sets

We have defined earlier what is meant by a subspace. We shall now describe a good way (indeed, probably the standard way) to specify subspaces.

Definition 1.12 Let V be a vector space over a field F and consider a finite set $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ of vectors in V. A *linear combination* of these vectors is a vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

for some $\alpha_1, \alpha_2, \dots, \alpha_k \in F$.

The set of all such linear combinations is called the *span* of the vectors v_1, v_2, $\dots$, v_k and is denoted by $\mathrm{Span}(v_1, v_2, \dots, v_k)$ or by $\mathrm{Span}(\mathscr{A})$.

Remarks

(i) We shall often use the familiar summation notation to abbreviate a linear combination:

$$\sum_{i=1}^{k} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k.$$

(ii) In some settings, some authors use the notation $\langle \mathscr{A} \rangle$ or $\langle v_1, v_2, \dots, v_k \rangle$ for the span of $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$. We will not do so in this course, since we wish to reserve angled brackets for inner products later in the module.

(iii) When $\mathscr{A}$ is an infinite set of vectors in a vector space V (over a field F), we need to apply a little care when defining $\mathrm{Span}(\mathscr{A})$. It does not make sense to add together infinitely many vectors because addition only allows us to combine two vectors at a time. Consequently for an arbitrary set $\mathscr{A}$ of vectors we make the following definition:

$$\mathrm{Span}(\mathscr{A}) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \;\middle|\; v_1, v_2, \dots, v_k \in \mathscr{A},\ \alpha_1, \alpha_2, \dots, \alpha_k \in F \right\}.$$

Thus $\mathrm{Span}(\mathscr{A})$ is the set of all linear combinations formed by selecting finitely many vectors from $\mathscr{A}$. When $\mathscr{A}$ is finite, this coincides with Definition 1.12.

Proposition 1.13 *Let $\mathscr{A}$ be a set of vectors in the vector space V. Then* $\mathrm{Span}(\mathscr{A})$ *is a subspace of V.*

Proof (Omitted in lectures): This proof was covered in *MT2501*.

We prove the proposition for the case when $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ is a finite set of vectors from V. The case when $\mathscr{A}$ is infinite requires few changes.

First note that, taking $\alpha_i = 0$ for each i, we see

$$\mathbf{0} = \sum_{i=1}^{k} 0 v_i \in \mathrm{Span}(\mathscr{A}).$$

Let $u, v \in \mathrm{Span}(\mathscr{A})$, say

$$u = \sum_{i=1}^{k} \alpha_i v_i \qquad \text{and} \qquad v = \sum_{i=1}^{k} \beta_i v_i$$

where the α_i and β_i are scalars. Then

$$u + v = \sum_{i=1}^{k} (\alpha_i + \beta_i) v_i \in \mathrm{Span}(\mathscr{A})$$

and if γ is a further scalar then

$$\gamma v = \sum_{i=1}^{k} (\gamma \alpha_i) v_i \in \mathrm{Span}(\mathscr{A}).$$

Thus $\mathrm{Span}(\mathscr{A})$ is a non-empty subset of V which is closed under addition and scalar multiplication; that is, it is a subspace of V. □

It is fairly easy to see that if W is a subspace of a vector space V, then $W = \mathrm{Span}(\mathscr{A})$ for some choice of $\mathscr{A} \subseteq W$. Indeed, the fact that W is closed under addition and scalar multiplication ensures that linear combinations of its elements are again in W and hence $W = \mathrm{Span}(W)$. However, what we will typically want to do is seek sets $\mathscr{A}$ which span particular subspaces where $\mathscr{A}$ can be made reasonably small.

Definition 1.14 A *spanning set* for a subspace W is some subset $\mathscr{A}$ of W such that $\mathrm{Span}(\mathscr{A}) = W$.

Thus if $\mathrm{Span}(\mathscr{A}) = W$, then each element of W can be written in the form

$$v = \sum_{i=1}^{k} \alpha_i v_i$$

where $v_1, v_2, \dots, v_k \in \mathscr{A}$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars from the base field. (Note that $\mathscr{A} \subseteq \mathrm{Span}(\mathscr{A})$, by definition, so if $W = \mathrm{Span}(\mathscr{A})$ then necessarily we must choose $\mathscr{A}$ to be a subset of W.)

We generally seek to find efficient choices of spanning sets $\mathscr{A}$; i.e., make $\mathscr{A}$ as small as possible.

Example 1.15 (i) Since every vector in $\mathbb{R}^3$ has the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we conclude that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a spanning set for $\mathbb{R}^3$. However, note that although this is probably the most natural spanning set, it is not the only one. For example,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x+y}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{x-y}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is also a spanning set.

We can also add vectors to a set that already spans and produce yet another spanning set. (Though there will inevitably be a level of redundancy to this and this relates to the concept of linear independence which we address next.) For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

is a spanning set for $\mathbb{R}^3$, since every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors appearing (in multiple ways); for example,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(x + \frac{z}{2} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (y+z) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{z}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{z}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

(ii) Recall the vector space $F[x]$ of polynomials over the field F; its elements have the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

We can therefore write

$$f(x) = a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) + \cdots + a_m f_m(x)$$

where $f_i(x) = x^i$ for $i = 0, 1, 2, \dots$. Hence the set

$$\mathscr{M} = \{1, x, x^2, x^3, \dots\}$$

of all *monomials* is a spanning set for $F[x]$.

Linearly independent sets and bases

In the previous section, we described spanning sets. Generally we like making these as efficient as possible and so we are led to consider linearly independent sets.

Definition 1.16 Let V be a vector space over a field F. A finite set $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ of vectors is called *linearly independent* if the only solution to the equation

$$\sum_{i=1}^{k} \alpha_i v_i = \mathbf{0}$$

(with $\alpha_i \in F$) is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

If $\mathscr{A}$ is not linearly independent, we shall call it *linearly dependent.*

The method to check whether a set of vectors is linearly independent was covered in *MT2501* and is fairly straightforward: Take the set of vectors $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ and consider the equation $\sum_{i=1}^{k} \alpha_i v_i = \mathbf{0}$. This should be viewed as a system of linear equations in the variables α_i. We apply the usual method of solving systems of linear equation, namely apply row operations (that is, *Gaussian elimination*) to the matrix associated to the system to determine what the solutions are. If $\alpha_i = 0$ for all i is the only solution then we will have shown $\mathscr{A}$ is linearly independent while if there are solutions which are not all zero then $\mathscr{A}$ is linearly dependent.

Example 1A *Determine whether the set $\{x + x^2, 1 - 2x^2, 3 + 6x\}$ is linearly independent in the vector space $\mathbb{R}[x]$ of all real polynomials.*

SOLUTION: We solve

$$\alpha(x + x^2) + \beta(1 - 2x^2) + \gamma(3 + 6x) = 0;$$

that is,

$$(\beta + 3\gamma) + (\alpha + 6\gamma)x + (\alpha - 2\beta)x^2 = 0. \tag{1.1}$$

Equating coefficients yields the system of equations

$$\begin{aligned} \beta + 3\gamma &= 0 \\ \alpha \qquad + 6\gamma &= 0 \\ \alpha - 2\beta \qquad &= 0; \end{aligned}$$

that is,

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 6 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A sequence of row operations (CHECK!) converts this to

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the original equation (1.1) is equivalent to

$$\begin{aligned} \alpha - 2\beta \qquad &= 0 \\ \beta + 3\gamma &= 0. \end{aligned}$$

Since there are fewer equations remaining than the number of variables, we have enough freedom to produce a *non-zero* solution. For example, if we set $\gamma = 1$, then $\beta = -3\gamma = -3$ and $\alpha = 2\beta = -6$. Hence the set $\{x + x^2, 1 - 2x^2, 3 + 6x\}$ is *linearly dependent.* □

If a set $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ of vectors in a vector space is linearly dependent, then there are solutions α_1, α_2, …, α_k to the equation

$$\sum_{i=1}^{k} \alpha_i v_i = \mathbf{0}$$

with not all $\alpha_i \in F$ equal to zero. Suppose that $\alpha_j \neq 0$ and rearrange the equation to

$$\alpha_j v_j = -(\alpha_1 v_1 + \cdots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \cdots + \alpha_k v_j).$$

Therefore

$$v_j = \left(-\tfrac{\alpha_1}{\alpha_j}\right) v_1 + \cdots + \left(-\tfrac{\alpha_{j-1}}{\alpha_j}\right) v_{j-1} + \left(-\tfrac{\alpha_{j+1}}{\alpha_j}\right) v_{j+1} + \cdots + \left(-\tfrac{\alpha_k}{\alpha_j}\right) v_k,$$

so v_j can be expressed as a linear combination of the other vectors in $\mathscr{A}$. Equally such an expression can be rearranged into an equation of linear dependence for $\mathscr{A}$. Hence we can make the following important observation:

Lemma 1.17 *Let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in the vector space V. Then $\mathscr{A}$ is linearly independent if and only if no vector in $\mathscr{A}$ can be expressed as a linear combination of the others.* $\square$

Suppose that $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ is a finite set of vectors which is *linearly dependent* and let $W = \mathrm{Span}(\mathscr{A})$. Now as $\mathscr{A}$ is linearly dependent, Lemma 1.17 tells us that one of the vectors in $\mathscr{A}$ is a linear combination of the others. Let us suppose that it is the vector v_j that is a such a linear combination, so

$$v_j = \alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \dots + \alpha_k v_k \tag{1.2}$$

for some scalars $\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_k$. Now if $w \in W$, then w is a linear combination of the vectors $v_1, v_2, \dots, v_k$. Substitute the formula (1.2) for the vector v_j and rearrange: the result is that w can also be expressed as a linear combination of the vectors $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k$.

We therefore deduce the following:

Lemma 1.18 *Let $\mathscr{A}$ be a linearly dependent set of vectors belonging to a vector space V. Then there is some vector v in $\mathscr{A}$ such that $\mathscr{A} \setminus \{v\}$ spans the same subspace as $\mathscr{A}$.* $\square$

As we started with a finite set $\mathscr{A}$, repeating the process described must eventually stop and at this point we will have produced a linearly independent set. Hence, we conclude:

Theorem 1.19 *Let V be a vector space. If $\mathscr{A}$ is a finite subset of V and $W = \mathrm{Span}(\mathscr{A})$, then there exists a linearly independent subset $\mathscr{B}$ with $\mathscr{B} \subseteq \mathscr{A}$ and $\mathrm{Span}(\mathscr{B}) = W$.* $\square$

Thus we can pass from a finite spanning set to a linearly independent spanning set by omitting the correct choice of vectors. Indeed, the above description tells us that we should omit a vector that can be expressed as a linear combination of the others, and then repeat. In particular, if the vector space V possesses a finite spanning set, then it possesses a linearly independent spanning set. Accordingly, we make the following definition:

Definition 1.20 Let V be a vector space over the field F. A *basis* for V is a linearly independent spanning set. We say that V is *finite-dimensional* if it possesses a finite spanning set; that is, if V possesses a finite basis. The *dimension* of V is the size of any basis for V and is denoted by $\dim V$.

Example 1.21 (i) The set

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $V = F^n$. We shall call it the *standard basis* for F^n. Hence $\dim F^n = n$ (as one would probably expect). Throughout we shall write

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the 1 is in the ith position, so that $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$.

[VERIFICATION (OMITTED IN LECTURES): If $\boldsymbol{v}$ is an arbitrary vector in V, say

$$\boldsymbol{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

(where $x_i \in F$). Thus $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ spans V. Suppose there exist scalars α_1, α_2, …, α_n such that

$$\sum_{i=1}^{n} \alpha_i \mathbf{e}_i = \mathbf{0};$$

that is,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Thus $\mathscr{B}$ is linearly independent.]

(ii) Let $\mathcal{P}_n$ be the set of polynomials over the field F of degree at most n:

$$\mathcal{P}_n = \{\, f(x) \mid f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \text{ for some } a_i \in F \,\}.$$

It is easy to check $\mathcal{P}_n$ is closed under addition and scalar multiplication, so $\mathcal{P}_n$ forms a vector subspace of the space $F[x]$ of all polynomials. The set of monomials $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathcal{P}_n$. Hence $\dim \mathcal{P}_n = n + 1$.

A basis in a vector space is useful since it enables us to express vectors in a unique way. In this sense, we can think of bases as the most efficient form of spanning sets.

Lemma 1.22 *Let V be a vector space of dimension n and let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for V. Then every vector in V can be expressed as a linear combination of the vectors in $\mathscr{B}$ in a* unique *way.*

PROOF: Let $v \in V$. Since $\mathscr{B}$ is a spanning set for V, we can certainly express v as a linear combination of the vectors in $\mathscr{B}$. Suppose we have two expressions for v:

$$v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i.$$

Hence

$$\sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = \mathbf{0}.$$

Since the set $\mathscr{B}$ is linearly independent, we deduce $\alpha_i - \beta_i = 0$ for all i; that is, $\alpha_i = \beta_i$ for all i. Hence our linear combination expression for v is indeed unique. $\square$

The dimension of a vector space is uniquely determined; that is, any two bases for a finite-dimensional vector space V have the same size. We shall deduce this fact from the following theorem, whose proof appears in the lecture notes but will be omitted in lectures.

Theorem 1.23 *Let V be a finite-dimensional vector space. Suppose that $\{v_1, v_2, \dots, v_m\}$ is a linearly independent set of vectors and $\{w_1, w_2, \dots, w_n\}$ is a spanning set for V. Then*

$$m \leqslant n.$$

Corollary 1.24 *Let V be a finite-dimensional vector space. Then any two bases for V have the same size and consequently $\dim V$ is uniquely determined.*

Proof: If $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ are bases for V, then they are both linearly independent and spanning sets for V, so Theorem 1.23 applied twice gives

$$m \leqslant n \qquad \text{and} \qquad n \leqslant m.$$

Hence $m = n$. □

Let us now turn to the proof of the theorem about linearly independent sets and spanning sets.

Proof of Theorem 1.23 (Omitted in lectures): Let V be a finite-dimensional vector space over a field F. We shall assume that $\{v_1, v_2, \dots, v_m\}$ is a linearly independent set in V and $\{w_1, w_2, \dots, w_n\}$ is a spanning set for V. In particular, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that

$$v_1 = \sum_{i=1}^{n} \alpha_i w_i.$$

Since $\{v_1, v_2, \dots, v_n\}$ is linearly independent, certainly $v_1 \neq \mathbf{0}$. Thus some of the α_i are non-zero. By re-arranging the w_i, there is no loss of generality in assuming $\alpha_1 \neq 0$. Then

$$w_1 = \frac{1}{\alpha_1} \left(v_1 - \sum_{i=2}^{n} \alpha_i w_i \right)$$

and this enables us to replace w_1 in any expression by a linear combination of the vectors v_1 and $w_2, \dots, w_n$. Since $V = \operatorname{Span}(w_1, w_2, \dots, w_n)$, we now deduce

$$V = \operatorname{Span}(v_1, w_2, \dots, w_n).$$

Suppose that we manage to show

$$V = \operatorname{Span}(v_1, v_2, \dots, v_j, w_{j+1}, \dots, w_n)$$

for some value of j where $j < m, n$. Then v_{j+1} is a vector in V, so can be expressed as a linear combination of $v_1, v_2, \dots, v_j, w_{j+1}, \dots, w_n$, say

$$v_{j+1} = \sum_{i=1}^{j} \beta_i v_i + \sum_{i=j+1}^{n} \beta_i w_i$$

for some scalars $\beta_i \in F$. Now if $\beta_{j+1} = \dots = \beta_n = 0$, then we would have

$$v_{j+1} = \sum_{i=1}^{j} \beta_i v_i,$$

which would contradict the fact that the set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent (see Lemma 1.17). Hence some β_i, with $i \geqslant j+1$, is non-zero. Re-arranging the w_i (again), we can assume that $\beta_{j+1} \neq 0$. Hence

$$w_{j+1} = \frac{1}{\beta_{j+1}} \left(v_{j+1} - \sum_{i=1}^{j} \beta_i v_i - \sum_{i=j+2}^{n} \beta_i w_i \right).$$

Consequently, we can replace w_{j+1} by a linear combination of the vectors v_1, v_2, $\ldots$, v_{j+1}, w_{j+2}, $\ldots$, w_n. Therefore

$$\begin{aligned} V &= \operatorname{Span}(v_1, v_2, \ldots, v_j, w_{j+1}, w_{j+2}, \ldots, w_n) \\ &= \operatorname{Span}(v_1, v_2, \ldots, v_{j+1}, w_{j+2}, \ldots, w_n). \end{aligned}$$

If it were the case that $m > n$, then this process stops when we have replaced all the w_i by v_i and have

$$V = \operatorname{Span}(v_1, v_2, \ldots, v_n).$$

But then v_{n+1} is a linear combination of v_1, v_2, $\ldots$, v_n, and this contradicts $\{v_1, v_2, \ldots, v_m\}$ being linearly independent.

Consequently $m \leqslant n$, as required. $\square$

In two examples that finish this section, we shall illustrate how to build bases for subspaces. In the first example, we start with a spanning set and, by omitting vectors that can be expressed as a linear combination of others in the set, reduce to a basis.

Example 1.25 *Let*

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ -6 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_5 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and let U be the subspace of $\mathbb{R}^4$ spanned by the set $\mathscr{A} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$. Find a basis $\mathscr{B}$ for U and hence determine the dimension of U.

Solution: Since $\dim \mathbb{R}^4 = 4$, the maximum size for a linearly independent set is 4 (by Theorem 1.23). This tells us that $\mathscr{A}$ is certainly not linearly independent; we need to find a linearly independent subset $\mathscr{B}$ of $\mathscr{A}$ that also spans U (see Theorem 1.19). This is done by finding which vectors in $\mathscr{A}$ can be written as a linear combination of the other vectors in $\mathscr{A}$ (see Lemma 1.17). We solve

$$\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 + \gamma \boldsymbol{v}_3 + \delta \boldsymbol{v}_4 + \varepsilon \boldsymbol{v}_5 = \boldsymbol{0}; \tag{1.3}$$

that is,

$$\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \\ -6 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ -1 & 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 3 & 0 & -6 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We apply row operations as follows to the appropriate augmented matrix:

$$
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & -1 & 0 \\
-1 & 1 & 3 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
3 & 0 & -6 & -1 & 0 & 0
\end{array}\right)
$$

$$
\longrightarrow \left(\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & -1 & 0 \\
0 & 3 & 3 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
0 & -6 & -6 & -1 & 3 & 0
\end{array}\right) \qquad \begin{array}{l} r_2 \mapsto r_2 + r_1 \\ r_4 \mapsto r_4 - 3r_1 \end{array}
$$

$$
\longrightarrow \left(\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
0 & 3 & 3 & 1 & 0 & 0 \\
0 & -6 & -6 & -1 & 3 & 0
\end{array}\right) \qquad r_2 \longleftrightarrow r_3
$$

$$
\longrightarrow \left(\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & -1 & -3 & 0
\end{array}\right) \qquad \begin{array}{l} r_3 \mapsto r_3 - 3r_2 \\ r_4 \mapsto r_4 + 6r_2 \end{array}
$$

$$
\longrightarrow \left(\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \qquad r_4 \mapsto r_4 + r_3
$$

So our equation (1.3) is equivalent to:

$$
\begin{aligned}
\alpha + 2\beta \qquad\quad - \varepsilon &= 0 \\
\beta + \gamma \qquad - \varepsilon &= 0 \\
\delta + 3\varepsilon &= 0
\end{aligned}
$$

Given arbitrary γ and ε, we can read off α, β and δ that solve the equation. Taking $\gamma = 1$, $\varepsilon = 0$ and $\gamma = 0$, $\varepsilon = 1$ tells us that the vectors $\boldsymbol{v}_3$ and $\boldsymbol{v}_5$ in $\mathscr{A}$ can be written as a linear combination of the others, as we shall now observe.

If $\gamma = 1$ and $\varepsilon = 0$, then the above tells us:

$$
\begin{aligned}
\delta &= -3\varepsilon = 0 \\
\beta &= -\gamma + \varepsilon = -1 \\
\alpha &= -2\beta + \varepsilon = 2
\end{aligned}
$$

Hence

$$2\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \boldsymbol{0},$$

so

$$\boldsymbol{v}_3 = -2\boldsymbol{v}_1 + \boldsymbol{v}_2. \tag{1.4}$$

If $\gamma = 0$ and $\varepsilon = 1$, then the above tells us:

$$
\begin{aligned}
\delta &= -3\varepsilon = -3 \\
\beta &= -\gamma + \varepsilon = 1 \\
\alpha &= -2\beta + \varepsilon = -1.
\end{aligned}
$$

Hence

$$-\boldsymbol{v}_1 + \boldsymbol{v}_2 - 3\boldsymbol{v}_4 + \boldsymbol{v}_5 = \mathbf{0},$$

so

$$\boldsymbol{v}_5 = \boldsymbol{v}_1 - \boldsymbol{v}_2 + 3\boldsymbol{v}_4. \tag{1.5}$$

Equations (1.4) and (1.5) tell us $\boldsymbol{v}_3, \boldsymbol{v}_5 \in \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4)$. Therefore any linear combination of the vectors in $\mathscr{A}$ can also be written as a linear combination of $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ (using (1.4) and (1.5) to achieve this). Hence

$$U = \operatorname{Span}(\mathscr{A}) = \operatorname{Span}(\mathscr{B}).$$

We finish by observing that $\mathscr{B}$ is linearly independent. Solve

$$\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2 + \gamma\boldsymbol{v}_4 = \mathbf{0};$$

that is,

$$\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

or

$$\begin{aligned} \alpha + 2\beta \qquad &= 0 \\ -\alpha \;\; + \beta + \gamma &= 0 \\ \beta \qquad &= 0 \\ 3\alpha \qquad - \gamma &= 0 \end{aligned}$$

In this case, we can automatically read off the solution:

$$\beta = 0, \qquad \alpha = -2\beta = 0, \qquad \gamma = 3\alpha = 0.$$

Hence $\mathscr{B}$ is linearly independent. It follows that $\mathscr{B}$ is the required basis for U. Then

$$\dim U = |\mathscr{B}| = 3. \qquad \square$$

Before the final example, we shall make some important observations concerning the creation of bases for finite-dimensional vector spaces.

Suppose that V is a finite-dimensional vector space, say $\dim V = n$, and suppose that we already have a linearly independent set of vectors, say $\mathscr{A} = \{v_1, v_2, \dots, v_m\}$. If $\mathscr{A}$ happens to span V, then it is a basis for V (and consequently $m = n$).

If not, there exists some vector, which we shall call v_{m+1}, such that $v_{m+1} \notin \operatorname{Span}(\mathscr{A})$. Consider the set

$$\mathscr{A}' = \{v_1, v_2, \dots, v_m, v_{m+1}\}.$$

Claim: $\mathscr{A}'$ is linearly independent.

Proof: Suppose

$$\sum_{i=1}^{m+1} \alpha_i v_i = \mathbf{0}.$$

If $\alpha_{m+1} \neq 0$, then

$$v_{m+1} = -\frac{1}{\alpha_{m+1}} \sum_{i=1}^{m} \alpha_i v_i \in \operatorname{Span}(\mathscr{A}),$$

which is a contradiction. Thus $\alpha_{m+1} = 0$, so

$$\sum_{i=1}^{m} \alpha_i v_i = \mathbf{0},$$

which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, as $\mathscr{A}$ is linearly independent. □

Hence, if $\mathscr{A}$ is a linearly independent subset which does not span V then we can adjoin another vector to produce a larger linearly independent set.

Let us now repeat the process. This cannot continue forever, since Theorem 1.23 says there is a maximum size for a linearly independent set, namely $n = \dim V$. Hence we must eventually reach a linearly independent set containing $\mathscr{A}$ that does span V. This proves:

Proposition 1.26 *Let V be a finite-dimensional vector space. Then every linearly independent set of vectors in V can be extended to a basis for V by adjoining a finite number of vectors.* □

Corollary 1.27 *Let V be a vector space of finite dimension n. If $\mathscr{A}$ is a linearly independent set containing n vectors, then $\mathscr{A}$ is a basis for V.*

PROOF: By Proposition 1.26, we can extend $\mathscr{A}$ to a basis $\mathscr{B}$ for V. But by Corollary 1.24, $\mathscr{B}$ contains $\dim V = n$ vectors. Hence we cannot have introduced any new vectors and so $\mathscr{B} = \mathscr{A}$ is the basis we have found. □

The final example of the section illustrates the process of extending a linearly independent set to produce a basis for a vector space.

Example 1.28 *Let $V = \mathbb{R}^4$. Show that the set*

$$\mathscr{A} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} \right\}$$

is a linearly independent set of vectors. Find a basis for $\mathbb{R}^4$ containing $\mathscr{A}$.

SOLUTION: To show $\mathscr{A}$ is linearly independent, we suppose

$$\alpha_1 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields four equations:

$$3\alpha_1 + \alpha_2 = 0, \quad \alpha_1 = 0, \quad 3\alpha_2 = 0, \quad 4\alpha_2 = 0.$$

Hence $\alpha_1 = \alpha_2 = 0$. Thus $\mathscr{A}$ is linearly independent.

We now seek to extend $\mathscr{A}$ to a basis of $\mathbb{R}^4$. We do so by first attempting to add the first vector of the standard basis for $\mathbb{R}^4$ to $\mathscr{A}$: Set

$$\mathscr{B} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Suppose

$$\alpha_1 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore

$$3\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 = 0, \quad 3\alpha_2 = 0, \quad 4\alpha_2 = 0.$$

So $\alpha_1 = \alpha_2 = 0$ (from the second and third equations) and we deduce $\alpha_3 = -3\alpha_1 - \alpha_2 = 0$. Hence our new set $\mathscr{B}$ is linearly independent.

If we now attempt to adjoin $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ to $\mathscr{B}$ and repeat the above, we would find that we were unable to prove the corresponding α_i are non-zero. Indeed,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Span}(\mathscr{B}).$$

Thus there is no need to adjoin the second standard basis vector to $\mathscr{B}$.

Now let us attempt to adjoin $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ to $\mathscr{B}$:

$$\mathscr{C} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Suppose

$$\alpha_1 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} 3\alpha_1 + \alpha_2 + \alpha_3 \qquad &= 0 \\ \alpha_1 \qquad\qquad\qquad &= 0 \\ 3\alpha_2 \qquad + \alpha_4 &= 0 \\ 4\alpha_2 \qquad\qquad &= 0 \end{aligned}$$

Therefore $\alpha_1 = \alpha_2 = 0$, from which we deduce $\alpha_3 = \alpha_4 = 0$. Thus we have produced a linearly independent set $\mathscr{C}$ of size 4. But $\dim \mathbb{R}^4 = 4$ and hence $\mathscr{C}$ must now be a basis for $\mathbb{R}^4$. $\square$

Chapter 2

Linear transformations

Linear transformations were discussed in detail in *MT2501 Linear Mathematics*. They are, in many ways, the central object of study in linear mathematics. In this chapter, we recall their definition and the most important observations made in *MT2501*. As with Chapter 1, proofs that were covered in the previous module will be omitted in the lectures. We shall also finish the chapter by giving a brief introducion to the space of linear maps between two spaces.

Definition and basic properties

The basic idea behind the definition of a linear transformation, which we now give, is that it is a function between two vector spaces that interacts well with the vector space structure (i.e., the addition and scalar multiplication).

Definition 2.1 Let V and W be vector spaces over the same field F. A *linear map* (also called a *linear transformation*) from V to W is a function $T\colon V \to W$ such that

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$, and

(ii) $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and $\alpha \in F$.

Comment: Sometimes we shall write Tv for the image of the vector v under the linear transformation T (instead of $T(v)$). We shall particularly do this when v is a column vector, so already possesses its own pair of brackets.

Over the course of the next few pages, we summarize the main properties that are important to recall from *MT2501*.

Lemma 2.2 *Let $T\colon V \to W$ be a linear map between two vector spaces over the field F. Then*

(i) $T(\mathbf{0}_V) = \mathbf{0}_W$*;*

(ii) $T(-v) = -T(v)$ *for all* $v \in V$*;*

(iii) *if* $v_1, v_2, \dots, v_k \in V$ *and* $\alpha_1, \alpha_2, \dots, \alpha_k \in F$*, then*

$$T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i T(v_i).$$

In the first part of this lemma, we write $\mathbf{0}_V$ and $\mathbf{0}_W$ for the zero vectors in the vector spaces V and W, respectively. We shall follow this convention in the start of this chapter. In due course, we shall drop the subscripts and use $\mathbf{0}$ to denote the zero vector in both spaces. The equation in part (i) will then be written $T(\mathbf{0}) = \mathbf{0}$. This should not cause confusion because the first one must be the zero vector in V since we can only apply T to vectors in the domain V, while the second must be in W since it is equal to the image of a vector under T so in the codomain W.

Proof (Omitted in lectures): This was covered in *MT2501*.

(i) Since $0 \cdot \mathbf{0}_V = \mathbf{0}_V$, it follows from the second condition in the definition that

$$T(\mathbf{0}_V) = T(0 \cdot \mathbf{0}_V) = 0 \cdot T(\mathbf{0}_V) = \mathbf{0}_W$$

(using Proposition 1.5(ii)).

(ii) Proposition 1.5(iv) tells us $(-1)v = -1v = -v$, so

$$T(-v) = T\left((-1)v\right) = (-1)T(v) = -T(v).$$

(iii) Using the two conditions of Definition 2.1, we deduce

$$T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} T(\alpha_i v_i) = \sum_{i=1}^{k} \alpha_i T(v_i).$$ □

The following two important subsets are associated to a linear map. They will both be used extensively in the rest of this module.

Definition 2.3 Let $T \colon V \to W$ be a linear transformation between vector spaces over a field F.

(i) The *image* of T is

$$T(V) = \operatorname{im} T = \{\, T(v) \mid v \in V \,\}$$

(which is a subset of W).

(ii) The *kernel*, or *null space*, of T is

$$\ker T = \{\, v \in V \mid T(v) = \mathbf{0}_W \,\}$$

(which is a subset of V).

The diagram in Figure 2.1 gives a schematic representation of the arrangement of the kernel and image as subsets of the spaces V and W in the above definition.

Note here that we are working with two vector spaces, each of which will possess its own zero vector. For emphasis in the definition, we are writing $\mathbf{0}_W$ for the zero vector belonging to the vector space W, so the kernel consists of those vectors in V which are mapped by T to the zero vector of W.

Of course, $T(v)$ has to be a vector in W, so actually there is little harm in writing simply $T(v) = \mathbf{0}$. For this equation to make any sense, the zero vector referred to must be that belonging to W, so confusion should not arise. Nevertheless to start with we shall write $\mathbf{0}_W$ just to be completely careful and clear.

Proposition 2.4 *Let $T \colon V \to W$ be a linear transformation between vector spaces V and W over the field F. The image and kernel of T are subspaces of W and V, respectively.*

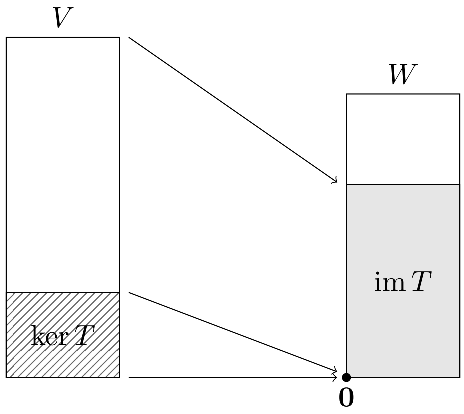


Figure 2.1: The kernel and image of a linear map $T\colon V \to W$

Proof (Omitted in lectures): This was covered in *MT2501*. Both parts of the proposition involve checking the conditions stated in Definition 1.6.

Certainly $\operatorname{im} T$ is non-empty since it contains all the images of vectors under the application of T. (Indeed, $T(\mathbf{0}_V) = 0_W$, so $\mathbf{0}_W$ is in the image of T.) Let $x, y \in \operatorname{im} T$. Then $x = T(u)$ and $y = T(v)$ for some $u, v \in V$. Hence

$$x + y = T(u) + T(v) = T(u + v) \in \operatorname{im} T$$

and

$$\alpha x = \alpha T(v) = T(\alpha v) \in \operatorname{im} T$$

for any $\alpha \in F$. Hence $\operatorname{im} T$ is a subspace of W.

Note that $T(\mathbf{0}_V) = \mathbf{0}_W$, so we see $\mathbf{0}_V \in \ker T$. So to start with $\ker T$ is non-empty. Now let $u, v \in \ker T$. Then

$$T(u + v) = T(u) + T(v) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$$

and

$$T(\alpha v) = \alpha T(v) = \alpha \cdot \mathbf{0}_W = \mathbf{0}_W.$$

Hence $u + v, \alpha v \in \ker T$ (for all $\alpha \in F$) and we deduce that $\ker T$ is a subspace of V. □

Definition 2.5 Let $T\colon V \to W$ be a linear transformation between vector spaces over the field F.

(i) The *rank* of T, which we shall denote $\operatorname{rank} T$, is the dimension of the image of T.

(ii) The *nullity* of T, which we shall denote $\operatorname{null} T$, is the dimension of the kernel of T.

Comment: The notations here are not uniformly established and I have simply selected a convenient notation rather than a definitive one. Many authors use different notation or, indeed, sometimes no specific notation whatsoever for these two concepts.

Theorem 2.6 (Rank-Nullity Theorem) *Let V and W be vector spaces over the field F with V finite-dimensional and let $T\colon V \to W$ be a linear transformation. Then*

$$\operatorname{rank} T + \operatorname{null} T = \dim V.$$

Comment: [For those who have done *MT2505 Abstract Algebra*] This can be viewed as an analogue of the First Isomorphism Theorem in the world of vector spaces. We shall not consider quotient spaces in this module, but if we did we would find that the dimension of the quotient $V/\ker T$ is equal to the different $\dim V - \dim\ker T$. Hence, upon rearranging the formula in the Rank-Nullity Theorem, we would be able to deduce

$$\dim(V/\ker T) = \dim\operatorname{im} T.$$

We shall see in Problem Sheet II, Question 3 that dimension essentially determines vector spaces; that is, two vector spaces of the same dimension over a field F are *isomorphic*. Hence

$$V/\ker T \cong \operatorname{im} T$$

which is what the First Isomorphism Theorem would say if it were to be established. Quotient spaces will, however, not be considered in this module and so this comment is purely to help you place the mathematics that you have met into context.

Proof (Omitted in lectures): This was covered in *MT2501.*

Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for $\ker T$ (so that $n = \operatorname{null} T$) and extend this (by Proposition 1.26) to a basis $\mathscr{C} = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$ for V (so that $\dim V = n + k$). We now seek to find a basis for $\operatorname{im} T$.

If $w \in \operatorname{im} T$, then $w = T(v)$ for some $v \in V$. We can write v as a linear combination of the vectors in the basis $\mathscr{C}$, say

$$v = \sum_{i=1}^{n+k} \alpha_i v_i.$$

Then, applying T and using linearity,

$$w = T(v) = T\left(\sum_{i=1}^{n+k} \alpha_i v_i\right) = \sum_{i=1}^{n+k} \alpha_i T(v_i) = \sum_{j=1}^{k} \alpha_{n+j} T(v_{n+j})$$

since $T(v_1) = \cdots = T(v_n) = \mathbf{0}$ as $v_1, \dots, v_n \in \ker T$. This shows that the set $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ spans $\operatorname{im} T$.

Now suppose that

$$\sum_{j=1}^{k} \beta_j T(v_{n+j}) = \mathbf{0};$$

that is,

$$T\left(\sum_{j=1}^{k} \beta_j v_{n+j}\right) = \mathbf{0}.$$

Hence $\sum_{j=1}^{k} \beta_j v_{n+j} \in \ker T$, so as $\mathscr{B} = \{v_1, \dots, v_n\}$ is a basis for $\ker T$, we have

$$\sum_{j=1}^{k} \beta_j v_{n+j} = \sum_{i=1}^{n} \gamma_i v_i$$

for some $\gamma_1, \gamma_2, \dots, \gamma_n \in F$. We now have an expression

$$(-\gamma_1)v_1 + \cdots + (-\gamma_n)v_n + \beta_1 v_{n+1} + \cdots + \beta_k v_{n+k} = \mathbf{0}$$

involving the vectors in the basis $\mathscr{C}$ for V. Since $\mathscr{C}$ is linearly independent, we conclude all the coefficients occurring here are zero. In particular, $\beta_1 = \beta_2 = \cdots = \beta_k = 0$. This shows that $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ is a linearly independent set and consequently a basis for $\operatorname{im} T$. Thus

$$\operatorname{rank}(T) = \dim \operatorname{im} T = k = \dim V - \operatorname{null} T$$

and this establishes the theorem. □

Constructing linear transformations

When you encounted linear transformations in *MT2501*, you will have met a number of examples of functions that were verified to be linear. However, there is a standard method to construct a linear map that was introduced in that module and which we shall now recall.

Proposition 2.7 *Let V and W be vector spaces over a field F. Suppose that $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V. Then if y_1, y_2, ..., y_n are any vectors in W, there is a unique linear map $T \colon V \to W$ such that*

$$T(v_i) = y_i \qquad \text{for } i = 1, 2, \dots, n.$$

Thus we uniquely define a linear map $T \colon V \to W$ by specifying where to map each of the basis vectors for V. The vectors y_1, y_2, ..., y_n appearing above are truly *arbitrary*. We do not need to be linearly independent or to span W; some can be the zero vector of W and we can even repeat the same vector many times over if we want.

We shall not verify the above proposition in the lectures (as it was covered in *MT2501*), but note that if $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V and $T \colon V \to W$ is a linear map that satisfies $T(v_i) = y_i$ for $i = 1$, 2, ..., n then we can determine $T(v)$ for every vector of V. Since $\mathscr{B}$ is a basis for V, a vector $v \in V$ can be uniquely expressed as

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

for some scalars α_i in F. Then linearity of T then implies

$$T(v) = T\Bigg(\sum_{i=1}^{n} \alpha_i v_i\Bigg) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \alpha_i y_i. \tag{2.1}$$

Hence T, if it exists, is uniquely specified by this formula (2.1). The proof of Proposition 2.7 then involves checking that this formula does define a linear map (that is, satisfies the conditions of Definition 2.1).

Proof of Proposition 2.7 (Omitted in lectures): We have just observed that if there is such a linear map $T \colon V \to W$ as in the statement of the proposition then it is given by the formula in Equation (2.1). Hence, if such a linear map T exists then it is unique. Let us now verify that this formula does define a linear transformation. Let $u, v \in V$, say

$$u = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{and} \qquad v = \sum_{i=1}^{n} \beta_i v_i$$

for some uniquely determined $\alpha_i, \beta_i \in F$. Then $u + v = \sum_{i=1}^{n} (\alpha_i + \beta_i) v_i$ and this must be the unique expression for $u + v$ in terms of the basis $\mathscr{B}$. So

$$T(u+v) = \sum_{i=1}^{n} (\alpha_i + \beta_i) y_i = \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \beta_i y_i = T(u) + T(v).$$

Similarly $\gamma u = \sum_{i=1}^{n} (\gamma\alpha_i) v_i$, so

$$T(\gamma u) = \sum_{i=1}^{n} (\gamma\alpha_i) y_i = \gamma \sum_{i=1}^{n} \alpha_i y_i = \gamma T(u) \qquad \text{for any } \gamma \in F.$$

This shows T is a linear transformation and completes the proof. □

We now give an example of our method of creating linear transformations.

Example 2.8 Define a linear transformation $T \colon \mathbb{R}^4 \to \mathbb{R}^3$ in terms of the standard basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ by

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \qquad T(\mathbf{e}_4) = \boldsymbol{y}_4 = \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix}.$$

Calculate the linear transformation T and its rank and nullity.

Solution: The effect of T on an arbitrary vector of $\mathbb{R}^4$ can be calculated by the linearity property:

$$\begin{aligned} T \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} &= T(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3 + \delta\mathbf{e}_4) \\ &= \alpha T(\mathbf{e}_1) + \beta T(\mathbf{e}_2) + \gamma T(\mathbf{e}_3) + \delta T(\mathbf{e}_4) \\ &= \alpha \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \delta \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 2\alpha - \beta - 5\delta \\ \alpha + \gamma - 2\delta \\ 3\alpha + \beta + 5\gamma - 5\delta \end{pmatrix}. \end{aligned}$$

[Exercise: Check by hand that this formula does really define a linear transformation $T \colon \mathbb{R}^4 \to \mathbb{R}^3$.]

Now let us determine the kernel of this transformation T. Suppose $v \in \ker T$. Here $\boldsymbol{v}$ is some vector in $\mathbb{R}^4$, say

$$\boldsymbol{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3 + \delta\mathbf{e}_4$$

where

$$T(\boldsymbol{v}) = \begin{pmatrix} 2\alpha - \beta - 5\delta \\ \alpha + \gamma - 2\delta \\ 3\alpha + \beta + 5\gamma - 5\delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have here three simultaneous equations in four variables which we convert to the matrix equation

$$\begin{pmatrix} 2 & -1 & 0 & -5 \\ 1 & 0 & 1 & -2 \\ 3 & 1 & 5 & -5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We solve this by performing the usual row operations used in Gaussian elimination:

$$\begin{aligned}
\left(\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 0 \\ 1 & 0 & 1 & -2 & 0 \\ 3 & 1 & 5 & -5 & 0 \end{array}\right) &\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 2 & -1 & 0 & -5 & 0 \\ 3 & 1 & 5 & -5 & 0 \end{array}\right) && r_1 \leftrightarrow r_2 \\
&\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right) && \begin{array}{l} r_2 \mapsto r_2 - 2r_1, \\ r_3 \mapsto r_3 - 3r_1 \end{array} \\
&\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) && \begin{array}{l} r_3 \mapsto r_3 + r_2, \\ r_2 \mapsto -r_2 \end{array}
\end{aligned}$$

So given arbitrary γ and δ, we require

$$\alpha + \gamma - 2\delta = 0 \qquad \text{and} \qquad \beta + 2\gamma + \delta = 0.$$

We remain with two degrees of freedom (the free choice of γ and δ) and so $\ker T$ is 2-dimensional:

$$\begin{aligned}
\ker T &= \left\{ \begin{pmatrix} -\gamma + 2\delta \\ -2\gamma - \delta \\ \gamma \\ \delta \end{pmatrix} \,\middle|\, \gamma, \delta \in \mathbb{R} \right\} \\
&= \left\{ \gamma \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \gamma, \delta \in \mathbb{R} \right\} \\
&= \operatorname{Span}\left(\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right).
\end{aligned}$$

It is easy to check these two spanning vectors are linearly independent (i.e., just observe that they are not scalar multiples of each other), so $\operatorname{null} T = \dim \ker T = 2$. The Rank-Nullity Theorem then says

$$\operatorname{rank} T = \dim \mathbb{R}^4 - \operatorname{null} T = 4 - 2 = 2.$$

Essentially this boils down to the four image vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4$ spanning a 2-dimensional space. Indeed, note that they are not linearly independent because

$$\begin{aligned}
\boldsymbol{y}_3 &= \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{y}_1 + 2\boldsymbol{y}_2 \\
\boldsymbol{y}_4 &= \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -2\boldsymbol{y}_1 + \boldsymbol{y}_2.
\end{aligned}$$

□

Further explanation of what we have observed in the above solution lies in the following result.

Proposition 2.9 *Let V be a finite-dimensional vector space over the field F with basis $\{v_1, v_2, \dots, v_n\}$ and let W be a vector space over F. Fix vectors $y_1, y_2, \dots, y_n$ in W and let $T\colon V \to W$ be the unique linear transformation given by $T(v_i) = y_i$ for $i = 1, 2, \dots, n$. Then*

(i) $\operatorname{im} T = \operatorname{Span}(y_1, y_2, \dots, y_n)$.

(ii) $\ker T = \{\mathbf{0}\}$ *if and only if* $\{y_1, y_2, \dots, y_n\}$ *is a linearly independent set.*

PROOF: (i) If $x \in \operatorname{im} T$, then $x = T(v)$ for some $v \in V$. We can write $v = \sum_{i=1}^{n} \alpha_i v_i$ for some $\alpha_i \in F$. Then

$$x = T(v) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \alpha_i y_i.$$

Thus, $\operatorname{im} T$ consists of all linear combinations of the vectors $y_1, y_2, \dots, y_n$; that is,

$$\operatorname{im} T = \operatorname{Span}(y_1, y_2, \dots, y_n).$$

(ii) This can be established by checking what it means for the vectors $y_1, y_2, \dots, y_n$, but we shall use the Rank-Nullity Theorem to reduce our work.

The Rank-Nullity Theorem tells us $\dim \ker T + \dim \operatorname{im} T = \dim V = n$. Hence $\ker T = \{\mathbf{0}\}$ if and only if $\dim \operatorname{im} T = n$. Since we have already observed $\{y_1, y_2, \dots, y_n\}$ is a spanning set for $\operatorname{im} T$, it follows that $\dim \operatorname{im} T = n$ if and only if $\{y_1, y_2, \dots, y_n\}$ is linearly independent. □

Example 2A *Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ in terms of the standard basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by*

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}.$$

Show that $\ker T = \{\mathbf{0}\}$ *and* $\operatorname{im} T = \mathbb{R}^3$.

SOLUTION: We check whether $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent. Solve

$$\alpha \boldsymbol{y}_1 + \beta \boldsymbol{y}_2 + \gamma \boldsymbol{y}_3 = \mathbf{0};$$

that is,

$$\begin{aligned} 2\alpha \ - \beta \qquad &= 0 \\ \alpha \qquad - \gamma &= 0 \\ -\alpha + 2\beta + 4\gamma &= 0. \end{aligned}$$

The second equation tells us that $\gamma = \alpha$ while the first says $\beta = 2\alpha$. Substituting for β and γ in the third equation gives

$$-\alpha + 4\alpha + 4\alpha = 7\alpha = 0.$$

Hence $\alpha = 0$ and consequently $\beta = \gamma = 0$.

This shows $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent. Consequently, $\ker T = \{\mathbf{0}\}$ by Proposition 2.9. The Rank-Nullity Theorem now says

$$\dim \operatorname{im} T = \dim \mathbb{R}^3 - \dim \ker T = 3 - 0 = 3.$$

Therefore $\operatorname{im} T = \mathbb{R}^3$ as it has the same dimension.

[Alternatively, since $\dim \mathbb{R}^3 = 3$ and $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent, this set must be a basis for $\mathbb{R}^3$ (see Corollary 1.27). Therefore, by Proposition 2.9(i),

$$\operatorname{im} T = \operatorname{Span}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3) = \mathbb{R}^3,$$

once again.] □

The following example (omitted in the lectures) returns to Example 2.8 and exploits the theory just described to compute a basis for the image of the linear map under consideration.

Example 2B *Let $T \colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined in terms of the standard basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ by*

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \qquad T(\mathbf{e}_4) = \boldsymbol{y}_4 = \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix}.$$

Find a basis for the image of T.

Solution: In Example 2.8 we observed that $\dim \operatorname{im} T = \operatorname{rank} T = 2$. We also know from Proposition 2.9 that

$$\operatorname{im} T = \operatorname{Span}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4),$$

so we conclude that $\operatorname{im} T$ has a basis $\mathscr{C}$ containing 2 vectors and that satisfies $\mathscr{C} \subseteq \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4\}$. Note that

$$\{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is linearly independent. Indeed if

$$\alpha \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then we deduce straightaway $\alpha = 0$ and then $\beta = 0$. We now have a linearly independent subset of $\operatorname{im} T$ of the right size to be a basis. Hence $\mathscr{C} = \{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ is a basis for $\operatorname{im} T$. □

The matrix of a linear transformation

Let $T\colon V \to W$ be a linear map between finite-dimensional vector spaces. Proposition 2.7 says that T is completely determined once we know the images of the vectors in some basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ for V. If we also have a basis for W available, then we can express each image $T(v_i)$ in terms of it. The matrix of the linear map T encodes this in a useful manner.

Definition 2.10 Let V and W be finite-dimensional vector spaces over the field F and let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_m\}$ be bases for V and W, respectively. If $T\colon V \to W$ is a linear transformation, let

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

express the image of the vector v_j under T as a linear combination of the basis $\mathscr{C}$ (for $j = 1, 2, \dots, n$). The $m \times n$ matrix $[\alpha_{ij}]$ is called the *matrix of T with respect to the bases $\mathscr{B}$ and $\mathscr{C}$*. We shall denote this by $\operatorname{Mat}(T)$ or, when we wish to be explicit about the dependence upon the bases $\mathscr{B}$ and $\mathscr{C}$, by $\operatorname{Mat}_{\mathscr{B},\mathscr{C}}(T)$.

In the special case of a linear transformation $T\colon V \to V$ and some fixed basis $\mathscr{B}$ for V, we shall speak of the *matrix of T with respect to the basis $\mathscr{B}$* to mean $\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T)$.

Note that the entries of the jth column of the matrix of T are:

$$\begin{matrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{matrix}$$

i.e., the jth column specifies the image of $T(v_j)$ by listing its coefficients when expressed as a linear combination of the basis vectors in $\mathscr{C}$.

It should be noted that the matrix of a linear transformation does very much depend upon the choices of bases. Accordingly it is much safer to employ the notation $\operatorname{Mat}_{\mathscr{B},\mathscr{C}}(T)$ and retain reference to the bases involved.

What does the matrix of a linear transformation actually represent? This question could be answered at great length and can get as complicated and subtle as one wants. The short answer is that if V and W are m- and n-dimensional vector spaces over a field F, then they "look like" F^m and F^n (formally, are *isomorphic* to these spaces, see Problem Sheet II, Question 3, for details). Then T maps vectors from V into W in the same way that the matrix $\operatorname{Mat}(T)$ maps vectors from F^m into F^n. (There is a technical formulation of what "in the same way" means here, but that goes way beyond the requirements of this course. It would, for example, result in the kernels of the two linear maps being of the same dimension, similarly for the images, etc.)

The following example illustrates the process of computing the matrix of a linear map with respect to various bases. You will observe that the process is quite straightforward when one uses the standard basis for the space of column vectors F^n but when one uses a "non-standard" basis more work is involved.

Example 2.11 Define a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ by the following formula:

$$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+4y \\ y \\ 2z+t \\ z+2t \end{pmatrix}.$$

Let $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ denote the standard basis for $\mathbb{R}^4$ and let $\mathscr{C}$ be the basis

$$\mathscr{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Determine the matrices $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$, $\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T)$ and $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

Solution: We calculate

$$T(\mathbf{e}_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e}_1$$

$$T(\mathbf{e}_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 4\mathbf{e}_1 + \mathbf{e}_2$$

$$T(\mathbf{e}_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = 2\mathbf{e}_3 + \mathbf{e}_4$$

$$T(\mathbf{e}_4) = T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \mathbf{e}_3 + 2\mathbf{e}_4.$$

So the matrix of T with respect to the basis $\mathscr{B}$ is

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

[We leave it as an exercise for the reader to check that $\mathscr{C}$ is indeed a basis for $\mathbb{R}^4$. Do this by showing it is linearly independent, i.e., the only solution to

$$\begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is $\alpha = \beta = \gamma = \delta = 0$.]

We shall calculate the matrices $\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T)$ and $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

$$
\begin{aligned}
T(\boldsymbol{v}_1) &= T\begin{pmatrix}2\\0\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\4\\2\end{pmatrix} = 2\mathbf{e}_1 + 4\mathbf{e}_3 + 2\mathbf{e}_4 \\
T(\boldsymbol{v}_2) &= T\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}4\\1\\-2\\-1\end{pmatrix} = 4\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3 - \mathbf{e}_4 \\
T(\boldsymbol{v}_3) &= T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} = 2\mathbf{e}_3 + \mathbf{e}_4 \\
T(\boldsymbol{v}_4) &= T\begin{pmatrix}3\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\1\\2\end{pmatrix} = 3\mathbf{e}_1 + \mathbf{e}_3 + 2\mathbf{e}_4.
\end{aligned}
$$

Hence

$$
\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T) = \begin{pmatrix}2 & 4 & 0 & 3\\ 0 & 1 & 0 & 0\\ 4 & -2 & 2 & 1\\ 2 & -1 & 1 & 2\end{pmatrix}.
$$

To find $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$, we need to express each $T(\boldsymbol{v}_j)$ in terms of the basis $\mathscr{C}$.

$$
\begin{aligned}
T(\boldsymbol{v}_1) = \begin{pmatrix}2\\0\\4\\2\end{pmatrix} &= -2\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 8\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\0\\0\\1\end{pmatrix} \\
&= -2\boldsymbol{v}_1 + 8\boldsymbol{v}_3 + 2\boldsymbol{v}_4 \\
T(\boldsymbol{v}_2) = \begin{pmatrix}4\\1\\-2\\-1\end{pmatrix} &= \frac{7}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} - 8\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \begin{pmatrix}3\\0\\0\\1\end{pmatrix} \\
&= \tfrac{7}{2}\boldsymbol{v}_1 + \boldsymbol{v}_2 - 8\boldsymbol{v}_3 - \boldsymbol{v}_4 \\
T(\boldsymbol{v}_3) = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} &= -\frac{3}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \begin{pmatrix}3\\0\\0\\1\end{pmatrix} \\
&= -\tfrac{3}{2}\boldsymbol{v}_1 + 5\boldsymbol{v}_3 + \boldsymbol{v}_4 \\
T(\boldsymbol{v}_4) = \begin{pmatrix}3\\0\\1\\2\end{pmatrix} &= -\frac{3}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\0\\0\\1\end{pmatrix} \\
&= -\tfrac{3}{2}\boldsymbol{v}_1 + 4\boldsymbol{v}_3 + 2\boldsymbol{v}_4.
\end{aligned}
$$

Hence

$$\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = \begin{pmatrix} -2 & 3\frac{1}{2} & -1\frac{1}{2} & -1\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 8 & -8 & 5 & 4 \\ 2 & -1 & 1 & 2 \end{pmatrix}.$$

□

Change of basis

The change of basis formula was covered in *MT2501* and will be very important for a lot of what we do. Accordingly, we shall spend a short while recalling how it works.

Suppose that we have available two bases $\mathscr{B}$ and $\mathscr{C}$ for the same vector space V. The change of basis formula describes the relationship between the matrices $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ and $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$ of T with respect to these two bases. (A similar description can be given for a linear transformation $V \to W$ with two bases $\mathscr{B}, \mathscr{B}'$ for V and two bases $\mathscr{C}, \mathscr{C}'$ for W. This would be more complicated, but essentially the same ideas apply.)

Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ be the two bases for V under consideration. As they are both bases, we may write each vector in them as a linear combination of the vectors in the other basis:

$$w_j = \sum_{k=1}^{n} \lambda_{kj} v_k \qquad \text{and} \qquad v_\ell = \sum_{i=1}^{n} \mu_{i\ell} w_i. \tag{2.2}$$

Let

$$P = [\lambda_{ij}] \qquad \text{and} \qquad Q = [\mu_{ij}]$$

be the matrices containing these coefficients. Note that we write the coefficients expressing w_j in terms of the vectors v_i *down the jth column* of the matrix P and similarly the coefficients for v_j down the jth column of Q. Since these matrices encode the switch from basis $\mathscr{C}$ to $\mathscr{B}$ and back again, it turns out that they are inverses of each other:

$$Q = P^{-1}$$

(See the proof of the theorem below for justification.)

Now let $T\colon V \to V$ be a linear map and let $A = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = [\alpha_{ij}]$. This means that

$$T(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i. \tag{2.3}$$

Now apply T to the first formula in Equation (2.2), use the linearity of T, and then substitute in (2.3) and the second equation in (2.2):

$$T(w_j) = \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj} \right) w_i$$

Hence the matrix $B = \mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = [\beta_{ij}]$ of T with respect to the basis $\mathscr{C}$ is given by

$$\beta_{ij} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj};$$

that is,

$$B = QAP = P^{-1}AP.$$

This is the formula in the change of basis theorem:

Theorem 2.12 *Let V be a vector space of dimension n over a field F and let $T\colon V \to V$ be a linear transformation. Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ be bases for V and let A and B be the matrices of T with respect to $\mathscr{B}$ and $\mathscr{C}$, respectively. Then there is an invertible matrix P such that*

$$B = P^{-1}AP.$$

Specifically, the (i,j)th entry of P is the coefficient of v_i when w_j is expressed as a linear combination of the basis vectors in $\mathscr{B}$.

Proof (Omitted in lectures): This proof will be omitted in the lectures. It contains the details of what was summarized in the discussion before the statement of the theorem. This theorem was established in *MT2501*.

Since $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ are both bases for V, we can write a vector in each one as a linear combination of the vectors in the other:

$$w_j = \sum_{k=1}^{n} \lambda_{kj} v_k \tag{2.4}$$

and

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} w_i \tag{2.5}$$

for some scalars $\lambda_{kj}, \mu_{i\ell} \in F$. Let $P = [\lambda_{ij}]$ and $Q = [\mu_{ij}]$ be the matrices whose entries are the coefficients appearing in these formulae.

Let us substitute (2.4) into (2.5):

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} \sum_{k=1}^{n} \lambda_{ki} v_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{ki}\mu_{i\ell} \right) v_k.$$

This must be the unique way of writing v_ℓ as a linear combination of the vectors in $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$. Thus

$$\sum_{i=1}^{n} \lambda_{ki}\mu_{i\ell} = \delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

(This $\delta_{k\ell}$ is called the *Kronecker delta*.) The left-hand side is the formula for matrix multiplication, so

$$PQ = I.$$

Hence P and Q are invertible matrices with $Q = P^{-1}$.

Now suppose that $T\colon V \to V$ is a linear transformation whose matrix with respect to the basis $\mathscr{B}$ is $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = [\alpha_{ij}]$. This means

$$T(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i \qquad \text{for } j = 1, 2, \dots, n. \tag{2.6}$$

To find $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$, apply T to (2.4):

$$T(w_j) = T\left(\sum_{k=1}^{n} \lambda_{kj} v_k \right)$$

$$\begin{aligned}
&= \sum_{k=1}^{n} \lambda_{kj} T(v_k) \\
&= \sum_{k=1}^{n} \lambda_{kj} \sum_{\ell=1}^{n} \alpha_{\ell k} v_\ell && \text{(from (2.6))} \\
&= \sum_{\ell=1}^{n} \sum_{k=1}^{n} \alpha_{\ell k} \lambda_{kj} \sum_{i=1}^{n} \mu_{i\ell} w_i && \text{(from (2.5))} \\
&= \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj} \right) w_i.
\end{aligned}$$

Hence $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = B = [\beta_{ij}]$ where

$$\beta_{ij} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj};$$

that is,

$$B = QAP = P^{-1}AP.$$ □

We shall illustrate the change of basis formula with an example. This example is concerned with a 2-dimensional vector space, principally chosen because calculating the inverse of a 2×2 matrix is much easier than doing so with one of larger dimension. However, for larger dimension vector spaces exactly the same method should be used.

Example 2.13 *Let V be a 2-dimensional vector space over $\mathbb{R}$ with basis $\mathscr{B} = \{v_1, v_2\}$. Let*

$$w_1 = 3v_1 - 5v_2, \qquad w_2 = -v_1 + 2v_2 \tag{2.7}$$

and $\mathscr{C} = \{w_1, w_2\}$. Define the linear transformation $T \colon V \to V$ by

$$\begin{aligned}
T(v_1) &= 16v_1 - 30v_2 \\
T(v_2) &= 9v_1 - 17v_2.
\end{aligned}$$

Find the matrix $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

Solution: The formula for T tells us that the matrix of T in terms of the basis $\mathscr{B}$ is

$$A = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} 16 & 9 \\ -30 & -17 \end{pmatrix}.$$

The formula (2.7) expresses the w_j in terms of the v_i. Hence, our change of basis matrix is

$$P = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Then

$$\det P = 3 \times 2 - (-1 \times -5) = 6 - 5 = 1,$$

so

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

So

$$\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = P^{-1}AP$$

$$
\begin{aligned}
&= \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 16 & 9 \\ -30 & -17 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ -10 & -6 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.
\end{aligned}
$$

We have diagonalised our linear transformation T. We shall discuss this topic in more detail later in these notes.

As a check, observe

$$
\begin{aligned}
T(w_2) &= T(-v_1 + 2v_2) \\
&= -T(v_1) + 2T(v_2) \\
&= -(16v_1 - 30v_2) + 2(9v_1 - 17v_2) \\
&= 2v_1 - 4v_2 \\
&= -2(-v_1 + 2v_2) = -2w_2,
\end{aligned}
$$

and similarly for $T(w_1)$. □

Example 2C *Let*

$$
\mathscr{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.
$$

(i) *Show that $\mathscr{B}$ is a basis for $\mathbb{R}^3$.*

(ii) *Write down the change of basis matrix from the standard basis $\mathscr{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\mathscr{B}$.*

(iii) *Let*

$$
A = \begin{pmatrix} -2 & -2 & -3 \\ 1 & 1 & 2 \\ -1 & -2 & -2 \end{pmatrix}
$$

and view A as a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$. Find the matrix of A with respect to the basis $\mathscr{B}$.

Solution: (i) We first establish that $\mathscr{B}$ is linearly independent. Solve

$$
\alpha \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};
$$

that is,

$$
\begin{aligned}
\beta + 2\gamma &= 0 \\
\alpha \qquad - \gamma &= 0 \\
-\alpha - \beta \qquad &= 0.
\end{aligned}
$$

Thus $\gamma = \alpha$ and the first equation yields $2\alpha + \beta = 0$. Adding the third equation now gives $\alpha = 0$ and hence $\beta = \gamma = 0$. This show $\mathscr{B}$ is linearly independent and it is therefore a basis for $\mathbb{R}^3$ since $\dim \mathbb{R}^3 = 3 = |\mathscr{B}|$.

(ii) We write each vector in $\mathscr{B}$ in terms of the standard basis

$$\begin{aligned}
\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} &= \phantom{2\mathbf{e}_1 -{}} \mathbf{e}_2 - \mathbf{e}_3 \\
\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} &= \mathbf{e}_1 \phantom{{}-\mathbf{e}_2} - \mathbf{e}_3 \\
\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} &= 2\mathbf{e}_1 - \mathbf{e}_2
\end{aligned}$$

and write the coefficients appearing down the columns of the change of basis matrix:

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

(iii) Theorem 2.12 says $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) = P^{-1}AP$ (as the matrix of A with respect to the standard basis is A itself). We first calculate the inverse of P via the usual row operation method:

$$\begin{aligned}
\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) &\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{array}\right) && r_3 \mapsto r_3 + r_1 \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{array}\right) && r_1 \leftrightarrow r_2 \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) && r_3 \mapsto r_3 + r_2 \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) && \begin{array}{l} r_1 \mapsto r_1 + r_3 \\ r_2 \mapsto r_2 - 2r_3 \end{array}
\end{aligned}$$

Hence

$$P^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

and so

$$\begin{aligned}
\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) &= P^{-1}AP \\
&= \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 & -3 \\ 1 & 1 & 2 \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -2 & -1 \\ 2 & 4 & 3 \\ -2 & -3 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.
\end{aligned}$$

□

Spaces of linear maps

We shall finish this chapter by observing that the collection of linear maps between two vector spaces also forms a vector space. We begin by defining the addition and scalar multiplication operations on linear maps.

Definition 2.14 Let V and W be vector spaces over a field F. Let S and T be linear maps $V \to W$ and α be a scalar from F.

(i) We define the *sum* $S + T\colon V \to W$ by

$$(S + T)(v) = S(v) + T(v).$$

(ii) We define the *scalar multiple* $\alpha T\colon V \to W$ by

$$(\alpha T)(v) = \alpha \cdot T(v).$$

Thus to compute the sum of two linear transformations $S, T\colon V \to W$, one applies these linear maps to a vector v from V and then sum the resulting vectors in W. Similarly for the scalar multiple of a linear map T: one applies T to a vector in V and then multiply the resulting vector from W by the scalar.

Lemma 2.15 *Let S and T be linear maps $V \to W$ between two vector spaces over some field F and let $\alpha \in F$. Then the sum $S + T$ and the scalar multiple αT are both linear maps.*

PROOF: Let $u, v \in V$ and $\lambda, \mu \in F$. Then

$$\begin{aligned}
(S + T)(\lambda u + \mu v) &= S(\lambda u + \mu v) + T(\lambda u + \mu v) \\
&= \lambda S(u) + \mu S(v) + \lambda T(u) + \mu T(v) \\
&= \lambda\big(S(u) + T(u)\big) + \mu\big(S(v) + T(v)\big) \\
&= \lambda(S + T)(u) + \mu(S + T)(v)
\end{aligned}$$

which shows that $S + T$ is a linear map. Similarly,

$$\begin{aligned}
(\alpha T)(\lambda u + \mu v) = \alpha \cdot T(\lambda u + \mu v) &= \alpha\big(\lambda T(u) + \mu T(v)\big) \\
&= \alpha\lambda T(u) + \alpha\mu T(v) = \lambda(\alpha T)(u) + \mu(\alpha T)(v)
\end{aligned}$$

which shows that αT is a linear map. (The latter calculation will probably be omitted in lectures as it is so similar to the first.) □

Now that we have addition and scalar multiplication operations on the collection of linear maps between two vector spaces V and W, we may view this collection as a vector space.

Definition 2.16 Let V and W be vector spaces over a field F. Define $\mathcal{L}(V, W)$ to be the set of linear transformations $V \to W$. We shall view $\mathcal{L}(V, W)$ as a vector space with respect to the addition and scalar multiplication given in Definition 2.14.

We shall not verify in the lectures that $\mathcal{L}(V, W)$ is indeed a vector space. This verification (which is now simply a case of checking the axioms listed in Definition 1.3) will be relegated to Problem Sheet II, Questions 10, but we do state the following theorem.

Theorem 2.17 *Let V and W be vector spaces over a field F. Then the set $\mathcal{L}(V,W)$ of linear maps $V \to W$ is a vector space over F with respect to the addition and scalar multiplication defined in Definition 2.14.*

Furthermore, if $\dim V = n$ and $\dim W = m$, then $\dim \mathcal{L}(V,W) = mn$.

When we recall the fact that we can associate a matrix to each linear transformation, the above observation does not seem surprising. If $\dim V = n$ and $\dim W = m$, then the matrix (with respect to some chosen bases) of a linear map $T\colon V \to W$ is an $m \times n$ matrix $A = \mathrm{Mat}(T)$. As a consequence, there is a one-one correspondence between the linear maps in $\mathcal{L}(V,W)$ and the space $\mathrm{M}_{m\times n}(F)$ of $m \times n$ matrices over F. (The latter space is also sometimes denoted $F^{m\times n}$.) It turns out that

$$\mathrm{Mat}(S+T) = \mathrm{Mat}(S) + \mathrm{Mat}(T) \qquad \text{and} \qquad \mathrm{Mat}(\alpha T) = \alpha\,\mathrm{Mat}(T)$$

for all linear maps $S, T\colon V \to W$ and scalars $\alpha \in F$. This would show that $\mathcal{L}(V,W)$ is isomorphic to the vector space of $m \times n$ matrices over F and hence $\dim \mathcal{L}(V,W) = mn$. (See Problem Sheet II, Question 11, for verification of these assertions.)

Dual space

Let V be a vector space over a field F. An important special case of the space $\mathcal{L}(V,W)$ just described arises when we take $W = F$. This is the dual space and it arises in a wide range of situations (for example, in algebra and in functional analysis on the one hand, but also in quantum mechanics on the other).

Note that the field F can be viewed as a vector space over itself. It can be viewed as the same as the space F^1 of column vectors of length 1, but where we omit the brackets. The addition is the original addition in the field and scalar multiplication is the original multiplication in the field.

Definition 2.18 Let V be a vector space over a field F.

(i) A *linear functional* is a linear map $f\colon V \to F$ (where F is viewed as a vector space over itself).

(ii) The *dual space* V^* is the space of all linear functionals $V \to F$ with addition and scalar multiplication given by

$$(f+g)(v) = f(v) + g(v) \qquad \text{and} \qquad (\alpha f)(v) = \alpha\, f(v)$$

for linear functionals $f, g\colon V \to F$, scalars $\alpha \in F$ and vectors $v \in V$.

Thus, the dual space V^* is just the same as what we called $\mathcal{L}(V,F)$ in the notation of Definition 2.16 and its addition and scalar multiplication are those given in Definition 2.14. Since $\dim F = 1$, the following therefore follows immediately from Theorem 2.17:

Theorem 2.19 *Let V be a vector space over a field F. Then the dual space V^* is a vector space over F and $\dim V^* = \dim V$.*

Indeed, one can construct a natural basis for the dual space V^* as follows. Suppose that $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V. For each $i = 1, 2, \dots, n$, let $f_i\colon V \to F$ be the unique linear map given by

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(This is guaranteed to define a linear map $V \to F$ by Proposition 2.7.) The resulting set of linear functionals $\{f_1, f_2, \ldots, f_n\}$ forms a basis for V^* called the *dual basis*. (See Problem Sheet II, Question 12, for verification.)

Chapter 3

Direct sums

In this chapter, we define an important concept that will be used throughout the chapters that follow.

Definition and basic properties

We start with the definition:

Definition 3.1 Let V be a vector space over a field F. We say that V is the *direct sum* of two subspaces U_1 and U_2, written $V = U_1 \oplus U_2$, if every vector in V can be expressed *uniquely* in the form $u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$.

Proposition 3.2 *Let V be a vector space and U_1 and U_2 be subspaces of V. Then $V = U_1 \oplus U_2$ if and only if the following conditions hold:*

(i) $V = U_1 + U_2$,

(ii) $U_1 \cap U_2 = \{\mathbf{0}\}$.

Comment: Many authors use these two conditions to *define* what is meant by a direct sum and then show it is equivalent to our "unique expression" definition.

PROOF: By definition, $U_1 + U_2 = \{\, u_1 + u_2 \mid u_1 \in U_1,\ u_2 \in U_2 \,\}$, so certainly every vector in V can be expressed in the form $u_1 + u_2$ (where $u_i \in U_i$) if and only if $V = U_1 + U_2$. We must show that condition (ii) corresponds to the uniqueness part.

So suppose $V = U_1 \oplus U_2$. Let $u \in U_1 \cap U_2$. Then $u = u + \mathbf{0} = \mathbf{0} + u$ expresses u as the sum of a vector in U_1 and a vector in U_2 in two ways. The uniqueness condition forces $u = \mathbf{0}$, so $U_1 \cap U_2 = \{\mathbf{0}\}$.

Conversely, suppose $U_1 \cap U_2 = \{\mathbf{0}\}$. Suppose $v = u_1 + u_2 = u_1' + u_2'$ are expressions for a vector v where $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$. Then

$$u_1 - u_1' = u_2' - u_2 \in U_1 \cap U_2,$$

so $u_1 - u_1' = u_2' - u_2 = \mathbf{0}$ and we deduce $u_1 = u_1'$ and $u_2 = u_2'$. Hence our expressions are unique, so (i) and (ii) together imply $V = U_1 \oplus U_2$. $\square$

Example 3.3 *Let $V = \mathbb{R}^3$ and let*

$$U_1 = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right) \qquad \textit{and} \qquad U_2 = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right).$$

Show that $V = U_1 \oplus U_2$.

SOLUTION: Let us solve

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We find

$$\alpha + 2\beta = \alpha + \beta + 3\gamma = \alpha + \gamma = 0.$$

Thus $\gamma = -\alpha$, so the second equation gives $\beta - 2\alpha = 0$; i.e., $\beta = 2\alpha$. Hence $5\alpha = 0$, so $\alpha = 0$ which implies $\beta = \gamma = 0$. Thus the three vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

are linearly independent and hence form a basis for $\mathbb{R}^3$. Therefore every vector in $\mathbb{R}^3$ can be expressed (uniquely) as

$$\left[\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] + \left[\gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right] = u_1 + u_2 \in U_1 + U_2.$$

So $\mathbb{R}^3 = U_1 + U_2$. If $\boldsymbol{v} \in U_1 \cap U_2$, then

$$\boldsymbol{v} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and we would have

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Linear independence forces $\alpha = \beta = \gamma = 0$. Hence $\boldsymbol{v} = \mathbf{0}$, so $U_1 \cap U_2 = \{\mathbf{0}\}$. Thus $\mathbb{R}^3 = U_1 \oplus U_2$. □

The link between a basis for V and a direct sum decomposition $V = U_1 \oplus U_2$ arose within the solution of the example above. We formalise this link in the following observation.

Proposition 3.4 *Let $V = U_1 \oplus U_2$ be a finite-dimensional vector space expressed as a direct sum of two subspaces. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for U_1 and U_2, respectively, then $\mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for V.*

PROOF: Let $\mathscr{B}_1 = \{u_1, u_2, \dots, u_m\}$ and $\mathscr{B}_2 = \{v_1, v_2, \dots, v_n\}$. If $v \in V$, then $v = x + y$ where $x \in U_1$ and $y \in U_2$. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ span U_1 and U_2, respectively, there exist scalars α_i and β_j such that

$$x = \alpha_1 u_1 + \cdots + \alpha_m u_m \qquad \text{and} \qquad y = \beta_1 v_1 + \cdots + \beta_n v_n.$$

Then

$$v = x + y = \alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 v_1 + \cdots + \beta_n v_n$$

and it follows that $\mathscr{B} = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ spans V.

Now suppose

$$\alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 v_1 + \cdots + \beta_n v_n = \mathbf{0}$$

for some scalars α_i, β_i. Put

$$x = \alpha_1 u_1 + \cdots + \alpha_m u_m \in U_1 \qquad \text{and} \qquad y = \beta_1 v_1 + \cdots + \beta_n v_n \in U_2.$$

Then $x + y = \mathbf{0}$ must be the unique decomposition of $\mathbf{0}$ produced by the direct sum $V = U_1 \oplus U_2$; that is, it must be $\mathbf{0} + \mathbf{0} = \mathbf{0}$. Hence

$$\alpha_1 u_1 + \cdots + \alpha_m u_m = x = \mathbf{0} \qquad \text{and} \qquad \beta_1 v_1 + \cdots + \beta_n v_n = y = \mathbf{0}.$$

Linear independence of $\mathscr{B}_1$ and $\mathscr{B}_2$ now give

$$\alpha_1 = \cdots = \alpha_m = 0 \qquad \text{and} \qquad \beta_1 = \cdots = \beta_n = 0.$$

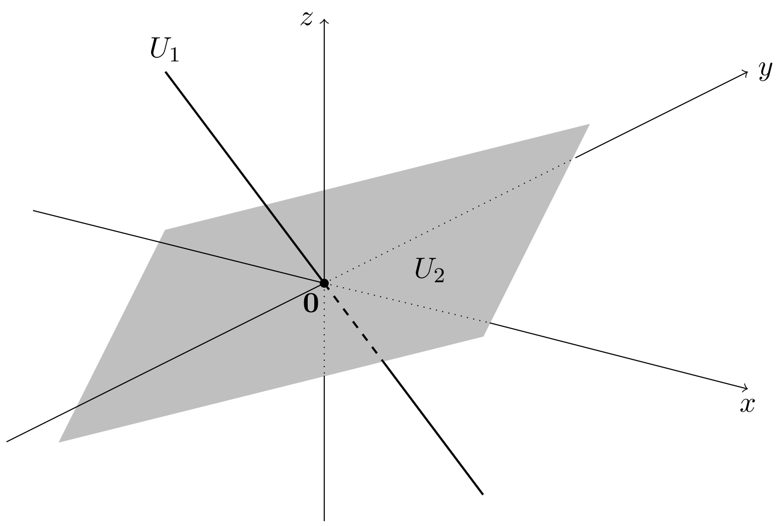
Hence $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is linearly independent and therefore a basis for V. □

Since the dimension of a vector space is equal to the size of a basis, we can immediately deduce the following fact about the dimension of a direct sum of two subspaces.

Corollary 3.5 *If $V = U_1 \oplus U_2$ is a finite-dimensional vector space expressed as a direct sum of two subspaces, then*

$$\dim V = \dim U_1 + \dim U_2.$$

Example 3.3 is in some sense typical of direct sums. To gain a visual understanding, the following picture illustrates the 3-dimensional space $\mathbb{R}^3$ as the direct sum of a 1-dimensional subspace U_1 and a 2-dimensional subspace U_2 (these being a line and a plane passing through the origin, respectively).



Projection maps

If we can decompose a vector space as a direct sum $V = U_1 \oplus U_2$ of two subspaces, then we can associate to this decomposition a pair of linear maps as follows:

Definition 3.6 Let $V = U_1 \oplus U_2$ be a vector space expressed as a direct sum of two subspaces. The two *projection maps* $P_1 \colon V \to V$ and $P_2 \colon V \to V$ onto U_1 and U_2, respectively, corresponding to this decomposition are defined as follows:

if $v \in V$, express v uniquely as $v = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$, then

$$P_1(v) = u_1 \qquad \text{and} \qquad P_2(v) = u_2.$$

Note that the uniqueness of expression guarantees that precisely one value is specified for $P_1(v)$ and one for $P_2(v)$. (If we only had $V = U_1 + U_2$ without $U_1 \cap U_2$ being the zero subspace, then we would have choice as to which expression $v = u_1 + u_2$ to use and we would not have well-defined maps.)

The basic properties of these projection maps are as follows:

Lemma 3.7 *Let $V = U_1 \oplus U_2$ be a direct sum of subspaces with projection maps $P_1 \colon V \to V$ and $P_2 \colon V \to V$. Then*

(i) *P_1 and P_2 are linear transformations;*

(ii) *$P_1(u) = u$ for all $u \in U_1$ and $P_1(w) = \mathbf{0}$ for all $w \in U_2$;*

(iii) *$P_2(u) = \mathbf{0}$ for all $u \in U_1$ and $P_2(w) = w$ for all $w \in U_2$;*

(iv) *$\ker P_1 = U_2$ and $\operatorname{im} P_1 = U_1$;*

(v) *$\ker P_2 = U_1$ and $\operatorname{im} P_2 = U_2$.*

One consequence of part (i) is that the functions P_1 and P_2 are elements in the vector space $\mathcal{L}(V, V)$ of linear maps $V \to V$. As a consequence, we may manipulate them using the addition and scalar multiplication operations defined on this space, as we shall do in Proposition 3.8 below.

Proof: We just deal with the parts relating to P_1. Those for P_2 are established by identical arguments. To simplify notation we shall discard the subscript and simply write P for the projection map onto U_1 associated to the direct sum decomposition $V = U_1 \oplus U_2$. This is defined by $P(v) = u_1$ when $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$.

(i) Let $v, v' \in V$ and write $v = u_1 + u_2$, $v' = u_1' + u_2'$ where $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$. Then

$$v + v' = (u_1 + u_1') + (u_2 + u_2')$$

and $u_1 + u_1' \in U_1$, $u_2 + u_2' \in U_2$. This must be the unique decomposition for $v + v'$, so

$$P(v + v') = u_1 + u_1' = P(v) + P(v').$$

Equally, if α is a scalar in the underlying field F, then $\alpha v = \alpha u_1 + \alpha u_2$ where $\alpha u_1 \in U_1$, $\alpha u_2 \in U_2$. Thus

$$P(\alpha v) = \alpha u_1 = \alpha P(v).$$

Hence P is a linear transformation.

(ii) If $u \in U_1$, then $u = u + \mathbf{0}$ is the decomposition we use to calculate the image of u under P, so $P(u) = u$.

If $w \in U_2$, then $w = \mathbf{0} + w$ is the required decomposition, so $P(w) = \mathbf{0}$.

(iv) For any vector v, $P(v)$ is always the U_1-part in the decomposition of v, so certainly $\operatorname{im} P \subseteq U_1$. On the other hand, if $u \in U_1$, then part (ii) says $u = P(u) \in \operatorname{im} P$. Hence $\operatorname{im} P = U_1$.

Part (ii) also says $P(w) = \mathbf{0}$ for all $w \in U_2$, so $U_2 \subseteq \ker P$. On the other hand, if $v = u_1 + u_2$ lies in $\ker P$, then $P(v) = u_1 = \mathbf{0}$, so $v = u_2 \in U_2$. Hence $\ker P = U_2$. $\square$

The major facts about projection maps are the following:

Proposition 3.8 *Let $P \colon V \to V$ be a projection corresponding to some direct sum decomposition of the vector space V. Then*

(i) $P^2 = P$;

(ii) $V = \ker P \oplus \operatorname{im} P$;

(iii) *$I - P$ is also a projection;*

(iv) $V = \ker P \oplus \ker(I - P)$.

Here, as usual, $I \colon V \to V$ denotes the identity transformation $I \colon v \mapsto v$ for $v \in V$. The formula $I - P$ refers to the addition and scalar multiplication operations on $\mathcal{L}(V, V)$ defined in Definition 2.14. Thus it is given by $(I - P)(v) = I(v) - P(v) = v - P(v)$ for all $v \in V$.

Proof: As a projection map, P is associated to a direct sum decomposition of V, so let us assume that $V = U_1 \oplus U_2$ and that $P = P_1$ is the corresponding projection onto the subspace U_1 (i.e., that P denotes the same projection as in the previous proof).

(i) If $v \in V$, then $P(v) \in U_1$, so by Lemma 3.7(ii),

$$P^2(v) = P(P(v)) = P(v).$$

Hence $P^2 = P$.

(ii) $\ker P = U_2$ and $\operatorname{im} P = U_1$, so

$$V = U_1 \oplus U_2 = \operatorname{im} P \oplus \ker P,$$

as required.

(iii) Let $Q \colon V \to V$ denote the projection onto U_2. If $v \in V$, say $v = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$, then

$$Q(v) = u_2 = v - u_1 = v - P(v) = (I - P)(v).$$

Hence $I - P$ is the projection Q.

(iv) $\ker P = U_2$, while $\ker(I - P) = \ker Q = U_1$. Hence

$$V = U_1 \oplus U_2 = \ker(I - P) \oplus \ker P. \qquad \square$$

We give an example to illustrate how projection maps depend on the choice of *both* summands in the direct sum decomposition.

Example 3.9 *Let*

$$U_1 = \operatorname{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right), \quad U_2 = \operatorname{Span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \quad \textit{and} \quad U_3 = \operatorname{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right).$$

Show that

$$\mathbb{R}^2 = U_1 \oplus U_2 \qquad \textit{and} \qquad \mathbb{R}^2 = U_1 \oplus U_3.$$

and, if $P \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto U_1 corresponding to the first decomposition and $Q \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto U_1 corresponding to the second decomposition, that $P \neq Q$.

Solution: If $\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$, then

$$\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$$

and

$$\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} = (x-y)\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix}.$$

Hence $\mathbb{R}^2 = U_1 + U_2 = U_1 + U_3$. Moreover,

$$U_1 = \left\{ \begin{pmatrix}x\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \qquad \text{and} \qquad U_2 = \left\{ \begin{pmatrix}0\\y\end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\},$$

so $U_1 \cap U_2 = \{\mathbf{0}\}$. Therefore we do have a direct sum $\mathbb{R}^2 = U_1 \oplus U_2$. It is similarly easy to see that $U_1 \cap U_3 = \{\mathbf{0}\}$, so the second sum is also direct.

We know by Lemma 3.7(ii) that

$$P(\boldsymbol{u}) = Q(\boldsymbol{u}) = \boldsymbol{u} \qquad \text{for all } \boldsymbol{u} \in U_1,$$

but if we take $\boldsymbol{v} = \begin{pmatrix}3\\2\end{pmatrix} \in \mathbb{R}^2$, we obtain different values for $P(\boldsymbol{v})$ and $Q(\boldsymbol{v})$. Indeed

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}$$

is the decomposition corresponding to $\mathbb{R}^2 = U_1 \oplus U_2$ which yields

$$P\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \in U_1$$

while

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix}$$

is the decomposition corresponding to $\mathbb{R}^2 = U_1 \oplus U_3$ which yields

$$Q\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \in U_1.$$

Also note $\ker P = U_2 \neq \ker Q = U_3$, which is more information indicating the difference between these two transformations. $\square$

Example 3A *Let $V = \mathbb{R}^3$ and $U = \mathrm{Span}(\boldsymbol{v}_1)$, where*

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

(i) *Find a subspace W such that $V = U \oplus W$.*

(ii) *Let $P\colon V \to V$ be the associated projection onto W. Calculate $P(\boldsymbol{u})$ where*

$$\boldsymbol{u} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

Solution: (i) We first extend $\{\boldsymbol{v}_1\}$ to a basis for $\mathbb{R}^3$. We claim that

$$\mathscr{B} = \left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. We solve

$$\alpha \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

that is,

$$3\alpha + \beta = -\alpha + \gamma = 2\alpha = 0.$$

Hence $\alpha = 0$, so $\beta = -3\alpha = 0$ and $\gamma = \alpha = 0$. Thus $\mathscr{B}$ is linearly independent. Since $\dim V = 3$ and $|\mathscr{B}| = 3$, we conclude that $\mathscr{B}$ is a basis for $\mathbb{R}^3$.

Let $W = \mathrm{Span}(\boldsymbol{v}_2, \boldsymbol{v}_3)$ where

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for V, if $\boldsymbol{v} \in V$, then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1) + (\alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3) \in U + W.$$

Hence $V = U + W$.

If $\boldsymbol{v} \in U \cap W$, then there exist $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\boldsymbol{v} = \alpha \boldsymbol{v}_1 = \beta_1 \boldsymbol{v}_2 + \beta_2 \boldsymbol{v}_3.$$

Therefore

$$\alpha \boldsymbol{v}_1 + (-\beta_1)\boldsymbol{v}_2 + (-\beta_2)\boldsymbol{v}_3 = \mathbf{0}.$$

Since $\mathscr{B}$ is linearly independent, we conclude $\alpha = -\beta_1 = -\beta_2 = 0$, so $\boldsymbol{v} = \alpha \boldsymbol{v}_1 = \mathbf{0}$. Thus $U \cap W = \{\mathbf{0}\}$ and so

$$V = U \oplus W.$$

(ii) We write $\boldsymbol{u}$ as a linear combination of the basis $\mathscr{B}$. Inspection shows

$$\begin{aligned}\boldsymbol{u} = \begin{pmatrix}4\\4\\4\end{pmatrix} &= 2\begin{pmatrix}3\\-1\\2\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 6\begin{pmatrix}0\\1\\0\end{pmatrix} \\ &= \begin{pmatrix}6\\-2\\4\end{pmatrix} + \begin{pmatrix}-2\\6\\0\end{pmatrix},\end{aligned}$$

where the first term in the last line belongs to U and the second to W. Hence

$$P(\boldsymbol{u}) = \begin{pmatrix}-2\\6\\0\end{pmatrix}$$

(since this is the W-component of $\boldsymbol{u}$). □

Direct sums of more summands

We finish the chapter by briefly describing the situation when V is expressed as a direct sum of more than two subspaces. The proof of the first proposition below is found on Problem Sheet III (see Question 8).

Definition 3.10 Let V be a vector space. We say that V is the *direct sum* of subspaces $U_1, U_2, \dots, U_k$, written $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$, if every vector in V can be *uniquely* expressed in the form $u_1 + u_2 + \dots + u_k$ where $u_i \in U_i$ for each i.

Again this can be translated into a condition involving sums and intersections, though the condition involving intersections is more complicated. We omit the proof.

Proposition 3.11 *Let V be a vector space with subspaces $U_1, U_2, \dots, U_k$. Then $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$ if and only if the following conditions hold:*

(i) $V = U_1 + U_2 + \dots + U_k$;

(ii) $U_i \cap (U_1 + \dots + U_{i-1} + U_{i+1} + \dots + U_k) = \{\mathbf{0}\}$ *for each* i.

We shall exploit the potential of direct sums to produce useful bases for our vector spaces. The following adapts quite easily from Proposition 3.4:

Proposition 3.12 *Let $V = U_1 \oplus U_2 \oplus \dots \oplus U_k$ be a direct sum of subspaces. If $\mathscr{B}_i$ is a basis for U_i for $i = 1, 2, \dots, k$, then $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \dots \cup \mathscr{B}_k$ is a basis for V.*

The direct sum of subspaces will arise explicitly when we discuss the orthogonal decomposition of an inner product space (see Theorem 6.16 below and the theory that follows it). Direct sum decomposition is also closely linked to the diagonalisation of linear maps and the Jordan normal form decomposition that are discussed in the next two chapters. (See, for example, Question 2 on Problem Sheet IV.)

Chapter 4

Diagonalisation of linear transformations

In this section, we seek to discuss the diagonalisation of a linear transformation; that is, to understand when a linear transformation can be represented by a diagonal matrix with respect to some basis. We begin by recalling material concerning eigenvectors and eigenvalues that was introduced in *MT2501*.

Eigenvectors and eigenvalues

Definition 4.1 Let V be a vector space over a field F and let $T\colon V \to V$ be a linear transformation. A *non-zero* vector v is an *eigenvector* for T with *eigenvalue* λ (where $\lambda \in F$) if

$$T(v) = \lambda v.$$

Note: The condition $v \neq \mathbf{0}$ is important since $T(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$ for *every* $\lambda \in F$, so considering $\mathbf{0}$ will never provide interesting information about our transformation T.

Note that $T(v) = \lambda v$ implies that $T(v) - \lambda v = \mathbf{0}$. Consequently, we make the following definition:

Definition 4.2 Let V be a vector space over a field F, let $T\colon V \to V$ be a linear transformation, and let $\lambda \in F$. The *eigenspace* corresponding to the eigenvalue λ is the subspace

$$\begin{aligned} E_\lambda = \ker(T - \lambda I) &= \{\, v \in V \mid T(v) - \lambda v = \mathbf{0} \,\} \\ &= \{\, v \in V \mid T(v) = \lambda v \,\}. \end{aligned}$$

(Recall that I denotes the identity transformation $v \mapsto v$.) Thus E_λ consists of all the eigenvectors of T with eigenvalue λ together with the zero vector $\mathbf{0}$. Note that $T - \lambda I$ is a linear transformation (by use of Lemma 2.15, since it is built from the linear transformations T and I), so E_λ is certainly a subspace of V.

From now on we assume that V is finite-dimensional over the field F. Thus in the discussion within the remainder of this chapter, V denotes a finite-dimensional vector space and $T\colon V \to V$ is a linear map.

Recall (see Problem Sheet II, Question 4) that a linear map $S\colon V \to V$ is invertible if and only if $\ker S = \{\mathbf{0}\}$. Let us write $A = \mathrm{Mat}(T)$ for the matrix of the linear map T with respect to some basis for V. Then $\lambda I - A$ is the matrix of $\lambda I - T$ with respect to the same basis and we observe:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } T \quad & \text{if and only if} \quad T(v) = \lambda v \text{ for some non-zero vector } v \in V \\ & \text{if and only if} \quad \ker(T - \lambda I) \neq \{\mathbf{0}\} \\ & \text{if and only if} \quad T - \lambda I \text{ is not invertible} \\ & \text{if and only if} \quad \det(\lambda I - A) = 0. \end{aligned}$$

In view of this observation, we make the following definition:

Definition 4.3 Let $T \colon V \to V$ be a linear transformation of the finite-dimensional vector space V (over F) and let A be the matrix of T with respect to some basis. The *characteristic polynomial* of T is

$$c_T(x) = \det(xI - A)$$

where x is an indeterminate variable.

Our observations above established the following lemma:

Lemma 4.4 *Suppose that $T \colon V \to V$ is a linear transformation of the finite-dimensional vector space V over F. Then λ is an eigenvalue of T if and only if λ is a root of the characteristic polynomial of T.* □

Remarks

(i) Some authors view it is as easier to calculate $\det(A - xI)$ rather than $\det(xI - A)$ and so use that formula as the definition of the characteristic polynomial. Since we are just multiplying every entry in the matrix by -1, the resulting determinant will merely be different by a factor of $(-1)^n = \pm 1$ (where $n = \dim V$) and the roots are unchanged. The reason for defining $c_T(x) = \det(xI - A)$ is that it ensures the highest degree term is *always* x^n (rather than $-x^n$ in the case that n is odd). The latter will be relevant in later discussions.

(ii) On the face of it the characteristic polynomial depends on the choice of basis, since, as we have seen, changing basis changes the matrix of a linear transformation. In fact, we get the same polynomial no matter which basis we use, as the following lemma observes.

Lemma 4.5 *Let V be a finite-dimensional vector space V over F and $T \colon V \to V$ be a linear transformation. The characteristic polynomial $c_T(x)$ is independent of the choice of basis for V.*

Consequently, $c_T(x)$ depends only on T.

Proof: Let A and B be the matrices of T with respect to two different bases for V. Theorem 2.12 tells us that $B = P^{-1}AP$ for some invertible matrix P. Then

$$P^{-1}(xI - A)P = xP^{-1}IP - P^{-1}AP = xI - B,$$

so

$$\begin{aligned} \det(xI - B) &= \det(P^{-1}(xI - A)P) \\ &= \det P^{-1} \cdot \det(xI - A) \cdot \det P \end{aligned}$$

$$= (\det P)^{-1} \cdot \det(xI - A) \cdot \det P$$
$$= \det(xI - A)$$

(since multiplication in the field F is commutative — see condition (v) of Definition 1.1). Hence we get the same answer for the characteristic polynomial. □

Diagonalisability

We may now move onto the diagonalisation of linear transformations.

Definition 4.6 (i) Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space V. We say that T is *diagonalisable* if there is a basis for V consisting of eigenvectors for T.

(ii) A square matrix A is *diagonalisable* if there is an invertible matrix P such that $P^{-1}AP$ is diagonal.

The two parts of this definition are simply two ways of looking at the same thing. If A is an $n \times n$ matrix, we may view it as a linear transformation $F^n \to F^n$ and forming $P^{-1}AP$ simply corresponds to choosing a (non-standard) basis for F^n with respect to which the transformation is represented by a diagonal matrix (see Theorem 2.12). Furthermore, the following observation links the existence of a basis of eigenvectors to the corresponding matrix being diagonalisable.

Proposition 4.7 *Let V be a finite-dimensional vector space, $T\colon V \to V$ be a linear transformation and $\mathscr{B}$ be a basis for V. Then the matrix of T with respect to $\mathscr{B}$ is diagonal if and only if the vectors in $\mathscr{B}$ are all eigenvectors for T.*

Proof (Omitted in lectures): This observation was made in *MT2501*.

Fix the basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ for V. If T is represented by a diagonal matrix with respect to $\mathscr{B}$, say

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$. Then $T(v_i) = \lambda_i v_i$ for $i = 1, 2, \dots, n$, so each basis vector in $\mathscr{B}$ is an eigenvector for T.

Conversely, if each vector in a basis $\mathscr{B}$ is an eigenvector for T (that is, if T is diagonalisable and $\mathscr{B}$ is the basis that witnesses this fact), then the matrix $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ is diagonal (with each diagonal entry being the corresponding eigenvalue). □

Example 4.8 *Let $V = \mathbb{R}^3$ and consider the linear transformation $T\colon V \to V$ given by the matrix*

$$A = \begin{pmatrix} 8 & 6 & 0 \\ -9 & -7 & 0 \\ 3 & 3 & 2 \end{pmatrix}$$

Show that T is diagonalisable, find a matrix P such that $D = P^{-1}AP$ is diagonal and find D.

Solution: To say that T is given by the above matrix is to say that this matrix represents T with respect to the standard basis for $\mathbb{R}^3$; i.e., that we obtain T by multiplying vectors on the left by A). We first calculate the characteristic polynomial:

$$\begin{aligned}\det(xI - A) &= \det \begin{pmatrix} x-8 & -6 & 0 \\ 9 & x+7 & 0 \\ -3 & -3 & x-2 \end{pmatrix} \\ &= (x-2)\left((x-8)(x+7) + 6 \times 9\right) \\ &= (x-2)\left((x-8)(x+7) + 54\right) \\ &= (x-2)(x^2 - x - 2) \\ &= (x-2)(x+1)(x-2) \\ &= (x+1)(x-2)^2,\end{aligned}$$

so

$$c_T(x) = (x+1)(x-2)^2$$

and the eigenvalues of T are -1 and 2. (We cannot yet guarantee we have enough eigenvectors to form a basis.)

We now need to go looking for eigenvectors. First we seek $\boldsymbol{v} \in \mathbb{R}^3$ such that $T(\boldsymbol{v}) = -\boldsymbol{v}$; i.e., such that $(T + I)(\boldsymbol{v}) = \mathbf{0}$. Thus we solve

$$\begin{pmatrix} 9 & 6 & 0 \\ -9 & -6 & 0 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$9x + 6y = 0, \qquad 3x + 3y + 3z = 0.$$

So given arbitrary x, take $y = -\frac{3}{2}x$ and $z = -x - y$. We have one degree of freedom (the choice of x) and we determine that

$$\begin{aligned}E_{-1} = \ker(T + I) &= \left\{ \begin{pmatrix} x \\ -\frac{3}{2}x \\ \frac{1}{2}x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left(\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right).\end{aligned}$$

Hence our eigenspace E_{-1} is one-dimensional and the vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is a suitable eigenvector with eigenvalue -1.

Now we seek $\boldsymbol{v} \in \mathbb{R}^3$ such that $T(\boldsymbol{v}) = 2\boldsymbol{v}$; i.e., $(T - 2I)(\boldsymbol{v}) = \mathbf{0}$. We therefore solve

$$\begin{pmatrix} 6 & 6 & 0 \\ -9 & -9 & 0 \\ 3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$x + y = 0.$$

Hence our eigenspace is

$$\begin{aligned} E_2 = \ker(T - 2I) &= \left\{ \begin{pmatrix} x \\ -x \\ z \end{pmatrix} \,\middle|\, x, z \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Hence the eigenspace E_2 is two-dimensional and we can find two linearly independent eigenvectors with eigenvalue 2, for example

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We conclude that with respect to the basis

$$\mathscr{B} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

the matrix of T is

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In particular, T is diagonalisable. (It is straightforward in this case to verify directly that $\mathscr{B}$ is linearly independent and hence it is a basis for the 3-dimensional space $\mathbb{R}^3$. It was also observed in *MT2501* that eigenvectors for distinct eigenvalues are linearly independent; see Proposition 4.13 below. This fact could also be used to show $\mathscr{B}$ is linearly independent.)

It remains to find P such that $D = P^{-1}AP$, but Theorem 2.12 tells us we need simply take the matrix whose entries are the coefficients of the vectors in $\mathscr{B}$ when expressed in terms of the standard basis:

$$P = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3,$$

so the entries of the first column are 2, -3 and 1, and similarly for the other columns. □

Before continuing to develop the theory of diagonalisation, we give an example that illustrates how these tools can be used in an applied mathematics setting. The following example is based on some found in Boas's textbook (see Examples 3 and 4 in [3, Section 3.12]).

Example 4.9 Solve the following system of differential equations involving differentiable functions x and y of one variable:

$$\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= 4x - 3y \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -2x + 3y \end{aligned}$$

SOLUTION: To simplify notation, we shall follow the common notational convention in applied mathematics and physics of denoting differentiation with respect to the variable t by a dot above the function. Define

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}.$$

Hence, we must solve the equation

$$\ddot{\boldsymbol{v}} = A\boldsymbol{v}$$

where $\ddot{\boldsymbol{v}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$. We begin by attempting to diagonalise the matrix A. The characteristic polynomial is

$$\begin{aligned} c_A(x) = \det(xI - A) = \det \begin{pmatrix} x-4 & 3 \\ 2 & x-3 \end{pmatrix} \\ &= (x-4)(x-3) - 6 \\ &= x^2 - 7x + 6 \\ &= (x-1)(x-6), \end{aligned}$$

so the eigenvalues of A are 1 and 6. Now

$$A - I = \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix} \qquad \text{and} \qquad A - 6I = \begin{pmatrix} -2 & -3 \\ -2 & -3 \end{pmatrix}$$

and we find the eigenvectors by solving $(A-I)\boldsymbol{v} = \mathbf{0}$ and $(A-6I)\boldsymbol{v} = \mathbf{0}$. The corresponding eigenspaces are

$$E_1 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \qquad \text{and} \qquad E_6 = \operatorname{Span}\left(\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right).$$

Then with respect to the basis $\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$, the transformation given by A is represented by the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

(It is a straightforward exercise to verify that $\mathscr{B}$ is linearly independent and hence a basis for $\mathbb{R}^2$. However, this will also be an immediately consequence of Proposition 4.13 below.) The change of basis matrix is

$$P = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$$

and then

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Define functions a and b by $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{w} = P^{-1}\boldsymbol{v}$. Substituting $\boldsymbol{v} = P\boldsymbol{w}$ into our vector differential equation $\ddot{\boldsymbol{v}} = A\boldsymbol{v}$ yields

$$P\ddot{\boldsymbol{w}} = AP\boldsymbol{w};$$

that is,

$$\begin{pmatrix} \ddot{a} \\ \ddot{b} \end{pmatrix} = \ddot{\boldsymbol{w}} = P^{-1}AP\boldsymbol{w} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence we must solve

$$\ddot{a} = a \qquad \text{and} \qquad \ddot{b} = 6b.$$

The standard methods for solving second-order differential equations show that

$$a(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-t} \qquad \text{and} \qquad b(t) = C\mathrm{e}^{\sqrt{6}t} + D\mathrm{e}^{-\sqrt{6}t}$$

for some constants A, B, C and D. We finally recover the functions $x(t)$ and $y(t)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{v} = P\boldsymbol{w} = \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - 3b \\ a + 2b \end{pmatrix};$$

that is,

$$\begin{aligned} x(t) &= a - 3b = A\mathrm{e}^{t} + B\mathrm{e}^{-t} - 3C\mathrm{e}^{\sqrt{6}t} - 3D\mathrm{e}^{-\sqrt{6}t} \\ y(t) &= a + 2b = A\mathrm{e}^{t} + B\mathrm{e}^{-t} + 2C\mathrm{e}^{\sqrt{6}t} + 2D\mathrm{e}^{-\sqrt{6}t}. \end{aligned}$$ □

Algebraic and geometric multiplicities

We shall now investigate when it is possible to find a basis of eigenvectors for a linear transformation; that is, what characterizes when a linear transformation is diagonalisable. The important question is how many linearly independent eigenvectors does one need to find for each eigenvalue and it is this question that we shall now address.

Suppose first that $T\colon V \to V$ is a diagonalisable linear transformation. Then there is a basis $\mathscr{B}$ for V with respect to which the matrix of T has the form

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ (possibly including repeats). The characteristic polynomial of T does not depend on the choice of basis (Lemma 4.5), so

$$\begin{aligned} c_T(x) = \det(xI - A) &= \det \begin{pmatrix} x - \lambda_1 & & & 0 \\ & x - \lambda_2 & & \\ & & \ddots & \\ 0 & & & x - \lambda_n \end{pmatrix} \\ &= (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n). \end{aligned}$$

So:

Lemma 4.10 *If the linear transformation $T\colon V \to V$ is diagonalisable, then the characteristic polynomial of T is a product of linear factors.* □

Here, a *linear* polynomial is one of degree 1, so we mean that the characteristic polynomial $c_T(x)$ is a product of factors of the form $\alpha x + \beta$. Of course, as the leading coefficient of $c_T(x)$ is x^n, the linear factors can always be arranged to have the form $x - \lambda$ with $\lambda \in F$. Then λ would be a root of $c_T(x)$ and hence an eigenvalue of T (by Lemma 4.4).

Note:

(i) This lemma only gives a necessary condition for diagonalisability. We shall next meet an example where $c_T(x)$ is a product of linear factors but T is not diagonalisable.

(ii) The field $\mathbb{C}$ of complex numbers is *algebraically closed*, i.e., every polynomial factorises as a product of linear factors. So Lemma 4.10 gives no information in that case.

Example 4.11 *Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix*

$$B = \begin{pmatrix} 8 & 3 & 0 \\ -18 & -7 & 0 \\ -9 & -4 & 2 \end{pmatrix}.$$

Determine whether T is diagonalisable.

Solution: Now

$$\begin{aligned} \det(xI - B) &= \det \begin{pmatrix} x-8 & -3 & 0 \\ 18 & x+7 & 0 \\ 9 & 4 & x-2 \end{pmatrix} \\ &= (x-2)\left((x-8)(x+7) + 3 \times 18\right) \\ &= (x-2)\left((x-8)(x+7) + 54\right) \\ &= (x-2)(x^2 - x - 2) \\ &= (x+1)(x-2)^2, \end{aligned}$$

so

$$c_T(x) = (x+1)(x-2)^2.$$

Again we have a product of linear factors, exactly as we did in Example 4.8. This time, however, issues arise when we seek to find eigenvectors with eigenvalue 2. Let us solve the equation $(T - 2I)(\boldsymbol{v}) = \mathbf{0}$; that is,

$$\begin{pmatrix} 6 & 3 & 0 \\ -18 & -9 & 0 \\ -9 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$6x + 3y = -18 - 9y = -9x - 4y = 0;$$

that is,

$$2x + y = 0, \qquad 9x + 4y = 0.$$

The first equation gives $y = -2x$, and when we substitute in the second we deduce $x = 0$ and so $y = 0$. Hence our eigenspace corresponding to eigenvalue 2 is

$$E_2 = \ker(T - 2I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Thus $\dim E_2 = 1$ and we cannot find two linearly independent eigenvectors with eigenvalue 2.

I shall omit the calculation but, by solving the equation $(T + I)(\boldsymbol{v}) = \mathbf{0}$, one finds the eigenspace with eigenvalue -1:

$$E_{-1} = \ker(T + I) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \right)$$

Consequently, one can only find a single linearly independent vector with eigenvalue -1. In conclusion, we cannot find a linearly independent set of eigenvectors for T containing more than two vectors. Hence there is no basis for $\mathbb{R}^3$ consisting of eigenvectors for T and T is not diagonalisable. □

The basic issue illustrated in this example is that not as many linearly independent eigenvectors could be found as there were relevant linear factors in the characteristic polynomial. Since $(x-2)^2$ occurs as a factor in the characteristic polynomial $c_T(x)$, it turns out that, in order to diagonalize T, we would need to find two linearly independent eigenvector with eigenvalue 2 (that is, we would need $\dim E_2 = 2$). To make this precise, we introduce the following two concepts:

Definition 4.12 Let V be a finite-dimensional vector space over the field F, let $T \colon V \to V$ be a linear transformation of V and let λ be an eigenvalue of T.

(i) The *algebraic multiplicity* of λ is the largest power k such that $(x - \lambda)^k$ is a factor of the characteristic polynomial $c_T(x)$.

(ii) The *geometric multiplicity* of λ is the dimension of the eigenspace E_λ corresponding to λ.

In theory, the above definitions could be applied to a scalar λ that is *not* an eigenvalue of T. However, if λ is not an eigenvalue of the linear map T then it is not a root of the characteristic polynomial $c_T(x)$ and there are no eigenvectors, so $E_\lambda = \{\mathbf{0}\}$. Hence, if λ is not an eigenvalue then its algebraic multiplicity and its geometric multiplicity would both equal 0. Consequently, we shall only be interested in these concepts when λ is an eigenvalue. In that case, λ is a root of the characteristic polynomial, so the algebraic multiplicity is at least 1, and the eigenspace is non-zero, so the geometric multiplicity is at least 1.

Our goal is to describe how these two multiplicities are linked. We shall make use of the following important fact (which appeared in *MT2501* and whose significance we have already indicated):

Proposition 4.13 *Let $T \colon V \to V$ be a linear transformation of a vector space V. A set of eigenvectors of T corresponding to* distinct *eigenvalues is linearly independent.*

PROOF: Let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be a set of eigenvectors of T. Let λ_i be the eigenvalue of v_i and assume that $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct. We proceed by induction on k.

If $k = 1$, then $\mathscr{A} = \{v_1\}$ consists of a single *non-zero* vector so is linearly independent.

So assume that $k > 1$ and that the result holds for smaller sets of eigenvectors for T. Suppose

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0}. \tag{4.1}$$

Apply T:

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_k T(v_k) = \mathbf{0},$$

so

$$\alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \cdots + \alpha_k\lambda_k v_k = \mathbf{0}.$$

Multiply (4.1) by λ_k and subtract:

$$\alpha_1(\lambda_1 - \lambda_k)v_1 + \alpha_2(\lambda_2 - \lambda_k)v_2 + \cdots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = \mathbf{0}.$$

This is an expression of linear dependence involving the eigenvectors v_1, v_2, $\dots$, v_{k-1}, so by induction

$$\alpha_i(\lambda_i - \lambda_k) = 0 \qquad \text{for } i = 1, 2, \dots, k-1.$$

By assumption the eigenvalues λ_i are distinct, so $\lambda_i - \lambda_k \neq 0$ for $i = 1$, 2, $\dots$, $k-1$, and, dividing by this non-zero scalar, we deduce

$$\alpha_i = 0 \qquad \text{for } i = 1, 2, \dots, k-1.$$

Equation 4.1 now yields $\alpha_k v_k = \mathbf{0}$, which forces $\alpha_k = 0$ as the eigenvector v_k is non-zero. This completes the induction step. $\square$

Theorem 4.14 *Let V be a finite-dimensional vector space over the field F and let $T \colon V \to V$ be a linear transformation of V.*

(i) *If the characteristic polynomial $c_T(x)$ is a product of linear factors (as always happens, for example, if $F = \mathbb{C}$), then the sum of the algebraic multiplicities equals* $\dim V$.

(ii) *Let $\lambda \in F$ and let r_λ be the algebraic multiplicity and n_λ be the geometric multiplicity of λ. Then*

$$n_\lambda \leqslant r_\lambda.$$

(iii) *The linear transformation T is diagonalisable if and only if $c_T(x)$ is a product of linear factors and $n_\lambda = r_\lambda$ for all eigenvalues λ.*

This theorem now explains Example 4.11 fully. It was an example where the characteristic polynomial splits into linear factors but the geometric multiplicity for one of the eigenvalues is smaller than its algebraic multiplicity.

PROOF: (i) Let $n = \dim V$ and write $c_T(x)$ as a product of linear factors

$$c_T(x) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \dots (x - \lambda_k)^{r_k}$$

where λ_1, λ_2, $\dots$, λ_k are the *distinct* roots of $c_T(x)$ (i.e., the distinct eigenvalues of T). Since $c_T(x)$ is the determinant of an $n \times n$ matrix, it is a polynomial of degree n, so

$$\dim V = n = r_1 + r_2 + \cdots + r_k,$$

the sum of the algebraic multiplicities.

(ii) Let λ be an eigenvalue of T and let $m = n_\lambda$, the dimension of the eigenspace $E_\lambda = \ker(T - \lambda I)$. Choose a basis $\{v_1, v_2, \dots, v_m\}$ for E_λ and extend to a basis $\mathscr{B} = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ for V. Note then

$$T(v_i) = \lambda v_i \qquad \text{for } i = 1, 2, \dots, m,$$

so the matrix of T with respect to $\mathscr{B}$ has the form

$$A = \operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & \lambda & \ddots & \vdots & * & \cdots & * \\ 0 & 0 & \ddots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \lambda & \vdots & & \vdots \\ \vdots & \vdots & & 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}$$

Hence

$$c_T(x) = \det \begin{pmatrix} x-\lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & x-\lambda & \ddots & \vdots & * & \cdots & * \\ 0 & 0 & \ddots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & x-\lambda & \vdots & & \vdots \\ \vdots & \vdots & & 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix} = (x-\lambda)^m p(x)$$

for some polynomial $p(x)$. (This polynomial $p(x)$ is equal to the determinant of the matrix obtained by taking $(n-m) \times (n-m)$ square at the bottom right of the above matrix.) Hence $r_\lambda \geqslant m = n_\lambda$.

(iii) Suppose that

$$c_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2} \dots (x-\lambda_k)^{r_k}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of T, so

$$r_1 + r_2 + \dots + r_k = n = \dim V$$

(by (i)). Let $n_i = \dim E_{\lambda_i}$ be the geometric multiplicity of λ_i. We suppose that $n_i = r_i$. Choose a basis $\mathscr{B}_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ for each E_{λ_i} and let

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \dots \cup \mathscr{B}_k = \{\, v_{ij} \mid i = 1, 2, \dots, k;\ j = 1, 2, \dots, n_i \,\}.$$

Claim: $\mathscr{B}$ is linearly independent.

Suppose

$$\sum_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n_i}} \alpha_{ij} v_{ij} = \mathbf{0}.$$

Let $w_i = \sum_{j=1}^{n_i} \alpha_{ij} v_{ij}$. Then w_i is a linear combination of vectors in the eigenspace E_{λ_i}, so $w_i \in E_{\lambda_i}$. Now

$$w_1 + w_2 + \dots + w_k = \mathbf{0}.$$

Proposition 4.13 says that eigenvectors for distinct eigenvalues are linearly independent, so the w_i cannot be eigenvectors. Therefore

$$w_i = \mathbf{0} \qquad \text{for } i = 1, 2, \dots, k;$$

that is,

$$\sum_{j=1}^{n_i} \alpha_{ij} v_{ij} = \mathbf{0} \qquad \text{for } i = 1, 2, \dots, k.$$

Since $\mathscr{B}_i$ is a basis for E_{λ_i}, it is linearly independent and we conclude that $\alpha_{ij} = 0$ for all i and j. Hence $\mathscr{B}$ is a linearly independent set.

Now since $n_i = r_i$ by assumption,

$$|\mathscr{B}| = n_1 + n_2 + \dots + n_k = n.$$

Hence $\mathscr{B}$ is a linearly independent set of size equal to the dimension of V. Therefore $\mathscr{B}$ is a basis for V and it consists of eigenvectors for T. Hence T is diagonalisable.

Conversely, suppose T is diagonalisable. We have already observed that $c_T(x)$ is a product of linear factors (Lemma 4.10). We may therefore maintain the notation of the first part of this proof. Since T is diagonalisable, there is a basis $\mathscr{B}$ for V consisting of eigenvectors for T. Let $\mathscr{B}_i = \mathscr{B} \cap E_{\lambda_i}$, that is, $\mathscr{B}_i$ consists of those vectors from $\mathscr{B}$ that have eigenvalue λ_i. As every vector in $\mathscr{B}$ is an eigenvector,

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \dots \cup \mathscr{B}_k.$$

As $\mathscr{B}$ is linearly independent, so is $\mathscr{B}_i$ and Theorem 1.23 tells us

$$|\mathscr{B}_i| \leqslant \dim E_{\lambda_i} = n_i.$$

Hence

$$n = |\mathscr{B}| = |\mathscr{B}_1| + |\mathscr{B}_2| + \dots + |\mathscr{B}_k| \leqslant n_1 + n_2 + \dots + n_k.$$

But $n_i \leqslant r_i$ and $r_1 + r_2 + \dots + r_k = n$, so we deduce $n_i = r_i$ for all i. This completes the proof of (iii). □

Example 4A *Let*

$$A = \begin{pmatrix} -1 & 2 & -1 \\ -4 & 5 & -2 \\ -4 & 3 & 0 \end{pmatrix}.$$

Show that A is not diagonalisable.

Solution: The characteristic polynomial of A is

$$\begin{aligned}
c_A(x) &= \det(xI - A) \\
&= \det \begin{pmatrix} x+1 & -2 & 1 \\ 4 & x-5 & 2 \\ 4 & -3 & x \end{pmatrix} \\
&= (x+1) \det \begin{pmatrix} x-5 & 2 \\ -3 & x \end{pmatrix} + 2 \det \begin{pmatrix} 4 & 2 \\ 4 & x \end{pmatrix} + \det \begin{pmatrix} 4 & x-5 \\ 4 & -3 \end{pmatrix} \\
&= (x+1)\big(x(x-5)+6\big) + 2(4x-8) + (-12 - 4x + 20) \\
&= (x+1)(x^2 - 5x + 6) + 8(x-2) - 4x + 8 \\
&= (x+1)(x-2)(x-3) + 8(x-2) - 4(x-2) \\
&= (x-2)\big((x+1)(x-3) + 8 - 4\big)
\end{aligned}$$

$$\begin{aligned} &= (x-2)(x^2-2x-3+4) \\ &= (x-2)(x^2-2x+1) \\ &= (x-2)(x-1)^2. \end{aligned}$$

In particular, the algebraic multiplicity of the eigenvalue 1 is 2.

We now determine the eigenspace for eigenvalue 1. We solve $(A-I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

We solve this by applying row operations:

$$\left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ -4 & 4 & -2 & 0 \\ -4 & 3 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad \begin{array}{l} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad r_1 \mapsto r_1 + 2r_3$$

So Equation (4.2) is equivalent to

$$-2x + z = 0 = -y + z.$$

Hence $z = 2x$ and $y = z = 2x$. Therefore the eigenspace is

$$E_1 = \left\{ \begin{pmatrix} x \\ 2x \\ 2x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right)$$

and we conclude $\dim E_1 = 1$. Thus the geometric multiplicity of 1 is not equal to the algebraic multiplicity, so A is not diagonalisable. □

Minimum polynomial

To gain further information about diagonalisation of linear transformations, we shall introduce the concept of the minimum polynomial (also called the minimal polynomial).

Let V be a vector space over a field F of dimension n. Consider a linear transformation $T\colon V \to V$. From observations made in Lemma 2.15 and in Question 7 on Problem Sheet II, we know the following facts about linear transformations:

(i) the composite of two linear transformations is also a linear transformation: in particular, T^2, T^3, T^4, ... are all linear transformations;

(ii) a scalar multiple of a linear transformation is a linear transformation;

(iii) the sum of two linear transformations is a linear transformation.

Consider a polynomial $f(x)$ over the field F, say

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k.$$

The facts above ensure that we have a well-defined linear transformation

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_k T^k.$$

This is what we shall mean by substituting the linear transformation T into the polynomial $f(x)$.

Moreover, since $\dim V = n$, the space $\mathcal{L}(V, V)$ of linear transformations $V \to V$ has dimension n^2 (see Theorem 2.17). Now if we return to our linear transformation $T \colon V \to V$, then let us consider the following collection of linear transformations

$$I,\ T,\ T^2,\ T^3,\ \ldots,\ T^{n^2}.$$

There are $n^2 + 1$ linear transformations listed, so they must form a linearly dependent set. Hence there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_{n^2} \in F$ such that

$$\alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_{n^2} T^{n^2} = 0$$

Omitting zero coefficients and dividing by the last non-zero scalar α_k yields an expression of the form

$$T^k + b_{k-1} T^{k-1} + \cdots + b_2 T^2 + b_1 T + b_0 I = 0$$

where $b_i = \alpha_i / \alpha_k$ for $i = 1, 2, \ldots, k-1$. Hence there exists a *monic* polynomial (that is, one whose leading coefficient is 1)

$$f(x) = x^k + b_{k-1} x^{k-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

such that

$$f(T) = 0.$$

We make the following definition:

Definition 4.15 Let $T \colon V \to V$ be a linear transformation of a finite-dimensional vector space over the field F. The *minimum polynomial* $m_T(x)$ is the *monic* polynomial over F of smallest degree such that

$$m_T(T) = 0.$$

Note: Our definition of the characteristic polynomial ensures that $c_T(x) = \det(xI - A)$ is always a monic polynomial.

We have observed that if V has dimension n and $T \colon V \to V$ is a linear transformation, then there certainly does exist some monic polynomial $f(x)$ such that $f(T) = 0$. Hence it makes sense to speak of *a* monic polynomial of smallest degree such that $f(T) = 0$. However, if

$$f(x) = x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_1 x + \alpha_0$$

and

$$g(x) = x^k + \beta_{k-1} x^{k-1} + \cdots + \beta_1 x + \beta_0$$

are two different polynomials of the same degree such that $f(T) = g(T) = 0$, then

$$h(x) = f(x) - g(x) = (\alpha_{k-1} - \beta_{k-1}) x^{k-1} + \cdots + (\alpha_1 - \beta_1) x + (\alpha_0 - \beta_0)$$

is a non-zero polynomial of smaller degree satisfying $h(T) = 0$, and some scalar multiple of $h(x)$ is then monic. We conclude that there is a *unique* monic polynomial $f(x)$ of smallest degree such that $f(T) = 0$.

We have also observed that if V has dimension n and $T \colon V \to V$, then there is a polynomial $f(x)$ of degree at most n^2 such that $f(T) = 0$. In fact, there is a major theorem that does considerably better:

Theorem 4.16 (Cayley–Hamilton Theorem) *Let $T \colon V \to V$ be a linear transformation of a finite-dimensional vector space V. If $c_T(x)$ is the characteristic polynomial of T, then*

$$c_T(T) = 0.$$

It takes a little while to prove this theorm and so we choose to omit the proof. To establish it, one needs to show that when T is substituted into

$$c_T(x) = \det(xI - A) \qquad \text{(where } A = \mathrm{Mat}(T)\text{)}$$

we produce the zero transformation. The main difficulty comes from the fact that x must be treated as a scalar when expanding the determinant and then T is substituted in instead of this scalar variable. Proofs can be found, for example, in Axler [1, 8.37] and in Blyth–Robertson [2, Theorem 10.1].

The upshot of the Cayley–Hamilton Theorem is to show that the minimum polynomial of T has degree at most the dimension of V and (as we shall see) to indicate close links between the minimum polynomial $m_T(x)$ and the characteristic polynomial $c_T(x)$.

To establish the relevance of the minimum polynomial to diagonalisation, we will need some basic properties of polynomials.

Facts about polynomials: Let F be a field and recall $F[x]$ denotes the set of polynomials with coefficients from F:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad \text{(where } a_i \in F\text{)}.$$

Then $F[x]$ is an example of what is known as a *Euclidean domain* (see *MT3505 Rings and Fields* for full details). A summary of some of its main properties are:

- We can add, multiply and subtract polynomials.
- If we attempt to divide $f(x)$ by $g(x)$ (where $g(x) \neq 0$), we obtain

  $$f(x) = g(x)q(x) + r(x)$$

  where either $r(x) = 0$ or the degree of $r(x)$ satisfies $\deg r(x) < \deg g(x)$ (i.e., we can perform long-division with polynomials).
- When the remainder is 0, that is, when $f(x) = g(x)q(x)$ for some polynomial $q(x)$, we say that $g(x)$ *divides* $f(x)$.

Those familiar with divisibility in the integers $\mathbb{Z}$ (particularly those who have attended *MT2505 Abstract Algebra*) will recognise these facts as being similar to properties of $\mathbb{Z}$ (which is also a standard example of a Euclidean domain).

Example 4.17 *Find the remainder upon dividing $x^3 + 2x^2 + 1$ by $x^2 + 1$ in the Euclidean domain $\mathbb{R}[x]$.*

SOLUTION:

$$\begin{aligned} x^3 + 2x^2 + 1 &= x(x^2 + 1) + 2x^2 - x + 1 \\ &= (x + 2)(x^2 + 1) - x - 1. \end{aligned}$$

Here the degree of the remainder $r(x) = -x - 1$ is less than the degree of $x^2 + 1$, so we have our required form. The quotient is $q(x) = x + 2$ and the remainder $r(x) = -x - 1$. □

Proposition 4.18 *Let V be a finite-dimensional vector space over a field F and let $T\colon V \to V$ be a linear transformation. If $f(x)$ is any polynomial (over F) such that $f(T) = 0$, then the minimum polynomial $m_T(x)$ divides $f(x)$.*

PROOF: Attempt to divide $f(x)$ by the minimum polynomial $m_T(x)$:

$$f(x) = m_T(x)q(x) + r(x)$$

for some polynomials $q(x)$ and $r(x)$ with either $r(x) = 0$ or $\deg r(x) < \deg m_T(x)$. Substituting the transformation T for the variable x gives

$$0 = f(T) = m_T(T)q(T) + r(T) = r(T)$$

since $m_T(T) = 0$ by definition. Since m_T has the smallest degree among non-zero polynomials which vanish when T is substituted, we conclude $r(x) = 0$. Hence

$$f(x) = m_T(x)q(x);$$

that is, $m_T(x)$ divides $f(x)$. □

Corollary 4.19 *Suppose that $T\colon V \to V$ is a linear transformation of a finite-dimensional vector space V. Then the minimum polynomial $m_T(x)$ divides the characteristic polynomial $c_T(x)$.*

PROOF: This follows immediately from the Cayley–Hamilton Theorem combined with Proposition 4.18. □

This corollary gives one link between the minimum polynomial and the characteristic polynomial. The following gives even stronger information, since it says a linear factor $(x-\lambda)$ occurs in one if and only if it occurs in the other.

Theorem 4.20 *Let V be a finite-dimensional vector space over a field F and let $T\colon V \to V$ be a linear transformation of V. Then the roots of the minimum polynomial $m_T(x)$ and the roots of the characteristic polynomial $c_T(x)$ coincide.*

Recall that the roots of $c_T(x)$ are precisely the eigenvalues of T. Thus this theorem has significance in the context of diagonalisation.

PROOF: Let λ be a root of $m_T(x)$, so $(x - \lambda)$ is a factor of $m_T(x)$. It then follows from Corollary 4.19 (i.e., from the Cayley–Hamilton Theorem) that $(x - \lambda)$ divides $c_T(x)$. (A direct proof, without using the Cayley–Hamilton Theorem, appears as Question 9 on Problem Sheet IV.)

Conversely, suppose λ is a root of $c_T(x)$, so λ is an eigenvalue of T. Hence there is an eigenvector v (note $v \neq \mathbf{0}$) with eigenvalue λ. Then $T(v) = \lambda v$,

$$T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v,$$

and in general, by a straightforward induction argument, $T^i(v) = \lambda^i v$ for all $i = 1, 2, \dots$. Suppose

$$m_T(x) = x^k + \alpha_{k-1}x^{k-1} + \dots + \alpha_1 x + \alpha_0.$$

Then

$$\mathbf{0} = m_T(T)v = (T^k + \alpha_{k-1}T^{k-1} + \dots + \alpha_1 T + \alpha_0 I)v$$

$$\begin{aligned}
&= T^k(v) + \alpha_{k-1}T^{k-1}(v) + \cdots + \alpha_1 T(v) + \alpha_0 v \\
&= \lambda^k v + \alpha_{k-1}\lambda^{k-1} v + \cdots + \alpha_1 \lambda v + \alpha_0 v \\
&= (\lambda^k + \alpha_{k-1}\lambda^{k-1} + \cdots + \alpha_1 \lambda + \alpha_0) v \\
&= m_T(\lambda) v.
\end{aligned}$$

Since $v \neq \mathbf{0}$, we conclude $m_T(\lambda) = 0$; i.e., λ is a root of $m_T(x)$. □

To see the full link to diagonalisability, the last result of this chapter that we prove is the following:

Theorem 4.21 *Let V be a finite-dimensional vector space over the field F and let $T \colon V \to V$ be a linear transformation. Then T is diagonalisable if and only if the minimum polynomial $m_T(x)$ is a product of distinct linear factors.*

PROOF: Suppose there is a basis $\mathscr{B}$ with respect to which T is represented by a diagonal matrix:

$$\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix}
\lambda_1 & & & & & & & & \\
& \ddots & & & & & & \mathbf{0} & \\
& & \lambda_1 & & & & & & \\
& & & \lambda_2 & & & & & \\
& & & & \ddots & & & & \\
& & & & & \lambda_2 & & & \\
& & & & & & \ddots & & \\
& \mathbf{0} & & & & & & \lambda_k & \\
& & & & & & & & \ddots \\
& & & & & & & & & \lambda_k
\end{pmatrix}$$

where the λ_i are the *distinct* eigenvalues. Then

$$A - \lambda_1 I = \begin{pmatrix}
0 & & & & & \\
& \ddots & & & & \\
& & 0 & & & \\
& & & \lambda_2 - \lambda_1 & & \\
& & & & \ddots & \\
& & & & & \lambda_k - \lambda_1
\end{pmatrix}$$

(with all non-diagonal entries being 0) and similar expressions apply to $A - \lambda_2 I, \dots, A - \lambda_k I$. Hence

$$(A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_k I) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0,$$

so

$$(T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_k I) = 0$$

Thus $m_T(x)$ divides $(x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_k)$ by Proposition 4.18. (In fact, by Theorem 4.20, it equals this product.) Hence $m_T(x)$ is a product of distinct linear factors.

To prove the converse, we shall make use of the following lemma:

Lemma 4.22 *Let U, V and W be finite-dimensional vector spaces and $T\colon U \to V$ and $S\colon V \to W$ be linear maps. Then*

$$\operatorname{null} ST \leqslant \operatorname{null} S + \operatorname{null} T.$$

Recall that the nullity $\operatorname{null} T$ of a linear map is the dimension of its kernel. The notation ST denotes the composite of the linear maps $T\colon V \to V$ followed by $S\colon V \to V$. We observed in Question 7 of Problem Sheet II that ST is a linear map.

PROOF: If $v \in \ker T$, then $ST(v) = S(\mathbf{0}) = \mathbf{0}$. Hence $\ker T \subseteq \ker ST$. Let $\{v_1, v_2, \dots, v_n\}$ be a basis for $\ker T$ and extend to a basis $\{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+m}\}$ for $\ker ST$. Now $T(v_{n+1})$, $T(v_{n+2})$, …, $T(v_{n+m})$ are vectors in V such that $S(T(v_{n+i})) = ST(v_i) = \mathbf{0}$ for $i = 1, 2, \dots, m$. Hence $\{T(v_{n+1}), T(v_{n+2}), \dots, T(v_{n+m})\}$ is a collection of vectors in the kernel of S. We shall show that this set of vectors is linearly independent.

Suppose $\sum_{i=1}^{m} \alpha_i T(v_{n+i}) = \mathbf{0}$; that is, by linearity of T,

$$T\left(\sum_{i=1}^{m} \alpha_i v_{n+i}\right) = \mathbf{0}.$$

Hence $\sum_{i=1}^{m} \alpha_i v_{n+i} \in \ker T$, so

$$\sum_{i=1}^{m} \alpha_i v_{n+i} = \sum_{j=1}^{n} \beta_j v_j$$

for some scalars β_1, β_2, …, β_n. However, by construction, $\{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+m}\}$ is linearly independent (as it is a basis for $\ker ST$). Hence $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ (and all β_j are zero also). Thus $\{T(v_{n+1}), T(v_{n+2}), \dots, T(v_{n+m})\}$ is a linearly independent subset of $\ker S$, so

$$m \leqslant \dim \ker S = \operatorname{null} S.$$

We conclude

$$\dim \ker ST = n + m = \dim \ker T + m \leqslant \dim \ker T + \dim \ker S;$$

that is,

$$\operatorname{null} ST \leqslant \operatorname{null} T + \operatorname{null} S. \qquad \square$$

We now return and complete the proof of Theorem 4.21. Let V be a finite-dimensional vector space, say $\dim V = n$. Let $T\colon V \to V$ be a linear map and suppose that the minimum polynomial $m_T(x)$ is a product of distinct linear factors

$$m_T(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_k).$$

Each λ_i is an eigenvalue of T (as, for example, observed in Theorem 4.20). Let r_i and n_i be the algebraic and geometric multiplicity of λ_i for $i = 1, 2, \dots, k$. Now

$$0 = m_T(T) = (T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_k I)$$

and $V = \ker m_T(T)$. Hence, by repeated application of Lemma 4.22,

$$\begin{aligned} n = \dim \ker m_T(T) &\leqslant \dim \ker(T - \lambda_1 I) + \dim \ker(T - \lambda_2 I) + \dots + \dim \ker(T - \lambda_k I) \\ &= n_1 + n_2 + \dots + n_k \end{aligned}$$

$$\begin{aligned} &\leqslant r_1 + r_2 + \cdots + r_k \\ &\leqslant \deg c_T(x) = n, \end{aligned}$$

by use of Theorem 4.14(ii) and the fact that the sum of the algebraic multiplicities r_i cannot be more than the degree of the characteristic polynomial. Since the left-hand and right-hand sides of the above inequalities are the same, it must therefore be the case that $r_i = n_i$ for $i = 1, 2, \dots, k$ and that $n = r_1 + r_2 + \cdots + r_k$. Hence $c_T(x) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \dots (x - \lambda_k)^{r_k}$ is a product of linear factors and Theorem 4.14(iii) then tells us that T is diagonalisable. □

Let us return to our two earlier examples and discuss them in the context of Theorem 4.21.

Example 4.23 *In Examples 4.8 and 4.11, we defined linear transformations* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by the matrices*

$$A = \begin{pmatrix} 8 & 6 & 0 \\ -9 & -7 & 0 \\ 3 & 3 & 2 \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} 8 & 3 & 0 \\ -18 & -7 & 0 \\ -9 & -4 & 2 \end{pmatrix},$$

respectively. Determine the minimum polynomials of A and B.

Solution: We observed that the matrices have the same characteristic polynomial

$$c_A(x) = c_B(x) = (x+1)(x-2)^2,$$

but A is diagonalisable while B is not. The minimum polynomial of each divides $(x+1)(x-2)^2$ and certainly has $(x+1)(x-2)$ as a factor (by Theorem 4.20). Now Theorem 4.21 tells us that $m_A(x)$ is a product of distinct linear factors, but $m_B(x)$ is not. Therefore

$$m_A(x) = (x+1)(x-2) \qquad \text{and} \qquad m_B(x) = (x+1)(x-2)^2.$$

[Exercise: Verify $(A+I)(A-2I) = 0$ and $(B+I)(B-2I) \neq 0$.] □

Example 4.24 *Consider the linear transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by the matrix*

$$D = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Calculate the characteristic polynomial of D, determine if D is diagonalisable and calculate the minimum polynomial.

Solution: The characteristic polynomial is

$$\begin{aligned} c_D(x) &= \det \begin{pmatrix} x-3 & 0 & -1 \\ -2 & x-2 & -2 \\ 1 & 0 & x-1 \end{pmatrix} \\ &= (x-3)(x-2)(x-1) + (x-2) \\ &= (x-2)(x^2 - 4x + 3 + 1) \\ &= (x-2)(x^2 - 4x + 4) \\ &= (x-2)^3. \end{aligned}$$

Therefore D is a diagonalisable only if $m_D(x) = x - 2$. But

$$D - 2I = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $m_D(x) \neq x - 2$. Thus D is not diagonalisable. Indeed

$$(D - 2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so we deduce $m_D(x) = (x-2)^2$. □

Example 4.25 *Consider the linear transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by the matrix*

$$E = \begin{pmatrix} -3 & -4 & -12 \\ 0 & -11 & -24 \\ 0 & 4 & 9 \end{pmatrix}.$$

Calculate the characteristic polynomial of E*, determine if* E *is diagonalisable and calculate its minimum polynomial.*

Solution:

$$\begin{aligned} c_E(x) &= \det \begin{pmatrix} x+3 & 4 & 12 \\ 0 & x+11 & 24 \\ 0 & -4 & x-9 \end{pmatrix} \\ &= (x+3)\left((x+11)(x-9) + 96\right) \\ &= (x+3)(x^2 + 2x - 3) \\ &= (x+3)(x-1)(x+3) \\ &= (x-1)(x+3)^2. \end{aligned}$$

So the eigenvalues of E are 1 and -3. Now E is diagonalisable only if its minimum polynomial $m_E(x)$ is equal to $(x-1)(x+3)$. We calculate

$$E - I = \begin{pmatrix} -4 & -4 & -12 \\ 0 & -12 & -24 \\ 0 & 4 & 8 \end{pmatrix}, \qquad E + 3I = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -8 & -24 \\ 0 & 4 & 12 \end{pmatrix},$$

so

$$(E - I)(E + 3I) = \begin{pmatrix} -4 & -4 & -12 \\ 0 & -12 & -24 \\ 0 & 4 & 8 \end{pmatrix} \begin{pmatrix} 0 & -4 & -12 \\ 0 & -8 & -24 \\ 0 & 4 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $m_E(x) = (x-1)(x+3)$ and E is diagonalisable. □

Example 4B *Let*

$$A = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 5 & 3 \\ -1 & -2 & 0 \end{pmatrix}.$$

Calculate the characteristic polynomial and the minimum polynomial of A*. Hence determine whether* A *is diagonalisable.*

SOLUTION:

$$\begin{aligned}
c_A &= \det(xI - A) \\
&= \det \begin{pmatrix} x & 2 & 1 \\ -1 & x-5 & -3 \\ 1 & 2 & x \end{pmatrix} \\
&= x \det \begin{pmatrix} x-5 & -3 \\ 2 & x \end{pmatrix} - 2 \det \begin{pmatrix} -1 & -3 \\ 1 & x \end{pmatrix} + \det \begin{pmatrix} -1 & x-5 \\ 1 & 2 \end{pmatrix} \\
&= x\bigl(x(x-5)+6\bigr) - 2(-x+3) + (-2-x+5) \\
&= x(x^2-5x+6) + 2(x-3) - x + 3 \\
&= x(x-3)(x-2) + 2(x-3) - (x-3) \\
&= (x-3)\bigl(x(x-2)+2-1\bigr) \\
&= (x-3)(x^2-2x+1) \\
&= (x-3)(x-1)^2.
\end{aligned}$$

Since the minimum polynomial divides $c_A(x)$ and has the same roots, we deduce

$$m_A(x) = (x-3)(x-1) \quad \text{or} \quad m_A(x) = (x-3)(x-1)^2.$$

We calculate

$$\begin{aligned}
(A-3I)(A-I) &= \begin{pmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -1 \\ 1 & 4 & 3 \\ -1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & -2 \\ -2 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix} \neq 0.
\end{aligned}$$

Hence $m_A(x) \neq (x-3)(x-1)$. We conclude

$$m_A(x) = (x-3)(x-1)^2.$$

This is not a product of distinct linear factors, so A is not diagonalisable. $\square$

Chapter 5

Jordan normal form

In the previous section we discussed at great length the diagonalisation of linear transformations. This is useful since it is much easier to work with diagonal matrices than arbitrary matrices. However, as we saw, not every linear transformation can be diagonalised. In this section, we discuss an alternative which, at least in the case of vector spaces over $\mathbb{C}$, can be used for any linear transformation or matrix.

Definition 5.1 A *Jordan block* is an $n \times n$ matrix of the form

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

for some positive integer n and some scalar λ; that is, the diagonal entries equal λ, the entries just above the diagonal equal 1 and all other entries are 0.

A linear transformation $T\colon V \to V$ (of a vector space V) has *Jordan normal form A* if there exists a basis $\mathscr{B}$ for V with respect to which the matrix of T is

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & J_{n_k}(\lambda_k) \end{pmatrix}$$

for some positive integers $n_1, n_2, \dots, n_k$ and scalars $\lambda_1, \lambda_2, \dots, \lambda_k$. (The occurrences of 0 here indicate zero matrices of appropriate sizes.)

Comment: Blyth & Robertson use the term "elementary Jordan matrix" for what we have called a Jordan block and use the term "Jordan block matrix" for something that is a hybrid between our two concepts above. I believe the terminology above is most common.

Theorem 5.2 *Let V be a finite-dimensional vector space and $T\colon V \to V$ be a linear transformation of V such that the characteristic polynomial $c_T(x)$ is a product of linear factors with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then there exists a basis for V with respect to which $\mathrm{Mat}(T)$ is in Jordan normal form where each Jordan block has the form $J_m(\lambda_i)$ for some m and some i.*

In particular, this theorem applies when our field is $\mathbb{C}$, since every polynomial is a product of linear factors over $\mathbb{C}$. When $c_T(x)$ is not a product of linear factors, Jordan normal form cannot be used. Instead, one uses something called *rational normal form*, which I shall not address here. (See, for example, [4, Section 11.4] for a discussion of rational normal form.)

Corollary 5.3 *Let A be a square matrix over $\mathbb{C}$. Then there exists an invertible matrix P (over $\mathbb{C}$) such that $P^{-1}AP$ is in Jordan normal form.*

This corollary follows from Theorem 5.2 and Theorem 2.12 (which tells us that change of basis corresponds to forming $P^{-1}AP$).

Corollary 5.4 *Let A and B be square matrices over $\mathbb{C}$. Then A and B are* similar *(that is, $B = P^{-1}AP$ for some invertible matrix P) if and only if they have the same Jordan normal form.*

According to Theorem 2.12, two matrices are similar when they represent the same linear map but with respect to different bases. This corollary tells us that over $\mathbb{C}$ (and, more generally, when the characteristic polynomials factorize as products of linear factors) we can determine similarity just using the Jordan normal form.

PROOF: If A and B have the same Jordan normal form J, then there exist invertible matrices P_1 and P_2 such that $P_1^{-1}AP_1 = P_2^{-1}BP_2 = J$. Then $P^{-1}AP = B$ where $P = P_1P_2^{-1}$. For the converse one needs to show that the Jordan normal form is essentially unique. (The details are omitted from this chapter but the ideas developed below could be adapted to verify this.) □

We shall not prove Theorem 5.2. It is reasonably complicated to prove and is most easily addressed by developing more advanced concepts and theory. Instead, we shall use this section to explain how to calculate the Jordan normal form associated to a linear transformation or matrix. We shall observe that we can determine the Jordan normal form of a linear map by knowing its eigenvalues, the algebraic and geometric multiplicities of these eigenvalues, and related information.

First consider a Jordan block

$$J = J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

We shall first determine its characteristic polynomial and minimum polynomial.

The characteristic polynomial of J is

$$c_J(x) = \det \begin{pmatrix} x-\lambda & -1 & & & \\ & x-\lambda & -1 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -1 \\ & & & & x-\lambda \end{pmatrix}$$

$$= (x-\lambda)\det\begin{pmatrix} x-\lambda & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & 0 & & x-\lambda \end{pmatrix}$$

$$\vdots$$

$$= (x-\lambda)^n.$$

When we turn to calculating the minimum polynomial, we note that $m_J(x)$ divides $c_J(x)$, so $m_J(x) = (x-\lambda)^k$ for some value of k with $1 \leqslant k \leqslant n$. Our problem is to determine what k must be.

We note

$$J - \lambda I = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Let us now calculate successive powers of $J - \lambda I$:

$$(J-\lambda I)^2 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & 0 \\ \vdots & \vdots & & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Repeatedly multiplying by $J - \lambda I$ successively moves the diagonal of 1s one level higher in the matrix. Thus

$$(J-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & 1 & 0 \\ \vdots & \vdots & & & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Finally, we find

$$(J-\lambda I)^{n-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad (J-\lambda I)^n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

So $(J-\lambda I)^n = 0$ but $(J-\lambda I)^{n-1} \neq 0$. Therefore

$$m_J(x) = (x-\lambda)^n.$$

In particular, we see the characteristic and minimum polynomials of a Jordan block coincide. We record these observations for future use:

Proposition 5.5 *Let $J = J_n(\lambda)$ be an $n \times n$ Jordan block. Then*

(i) $c_J(x) = (x-\lambda)^n$;

(ii) $m_J(x) = (x-\lambda)^n$;

(iii) *the eigenspace E_λ of J has dimension* 1.

PROOF: It remains to prove part (iii). To find the eigenspace E_λ, we solve $(J-\lambda I)(v) = \mathbf{0}$. We have calculated $J - \lambda I$ above, so we solve

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

that is,

$$\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Hence $x_2 = x_3 = \cdots = x_n = 0$, while x_1 may be arbitrary. Therefore

$$E_\lambda = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, x \in F \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right).$$

Hence $\dim E_\lambda = 1$, as claimed. □

We now use the information obtained in the previous proposition to tell us how to embark on solving the following general proposition.

Problem: Let V be a finite-dimensional vector space and let $T\colon V \to V$ be a linear transformation. If the characteristic polynomial $c_T(x)$ is a product of linear factors, find a basis $\mathscr{B}$ with respect to which T is in Jordan normal form and determine what this Jordan normal form is.

If $\mathscr{B}$ is the basis solving this problem, then

$$\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & 0 & \\ & 0 & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{pmatrix},$$

where $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, ..., $J_{n_k}(\lambda_k)$ are the Jordan blocks. When we calculate the characteristic polynomial $c_T(x)$ using this matrix, each block $J_{n_i}(\lambda_i)$ contributes a factor of $(x - \lambda_i)^{n_i}$ (see Proposition 5.5(i)). Collecting all the factors corresponding to the same eigenvalue, we conclude:

Observation 5.6 *The algebraic multiplicity of λ as an eigenvalue of T equals the sum of the sizes of the Jordan blocks $J_n(\lambda)$ (associated to λ) occurring in the Jordan normal form for T.*

This means, of course, that the number of times that λ occurs on the diagonal in the Jordan normal form matrix A is precisely the algebraic multiplicity r_λ of λ.

If particular, if $r_\lambda = 1$, a single 1×1 Jordan block occurs in A, namely $J_1(\lambda) = (\lambda)$. If $r_\lambda = 2$, then either two 1×1 Jordan blocks occur or a 2×2 Jordan block $J_2(\lambda)$ occurs in A. Thus A either contains

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Similar observations may be made for other small values of r_λ, but the possibilities grow more complicated as r_λ increases.

To distinguish between these possibilities, we first make use of the minimum polynomial. To ensure the block $J_{n_i}(\lambda_i)$ becomes 0 when we substitute A into the polynomial, we must have at least a factor $(x - \lambda_i)^{n_i}$ (see Proposition 5.5(ii)). Consequently:

Observation 5.7 *If λ is an eigenvalue of T, then the power of $(x - \lambda)$ occurring in the minimum polynomial $m_T(x)$ is $(x - \lambda)^m$ where m is the largest size of a Jordan block associated to λ occurring in the Jordan normal form for T.*

Observations 5.6 and 5.7 are enough to determine the Jordan normal form in small cases.

Example 5.8 *Let $V = \mathbb{R}^4$ and let $T\colon V \to V$ be the linear transformation given by the matrix*

$$B = \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 2 & 0 & 4 \\ 4 & 5 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Determine the Jordan normal form of T.

SOLUTION: We first determine the characteristic polynomial of T:

$$\begin{aligned}
c_T(x) &= \det \begin{pmatrix} x-2 & -1 & 0 & 3 \\ 0 & x-2 & 0 & -4 \\ -4 & -5 & x+2 & -1 \\ 0 & 0 & 0 & x+2 \end{pmatrix} \\
&= (x+2) \det \begin{pmatrix} x-2 & -1 & 0 \\ 0 & x-2 & 0 \\ -4 & -5 & x+2 \end{pmatrix} \\
&= (x+2)^2 \det \begin{pmatrix} x-2 & -1 \\ 0 & x-2 \end{pmatrix} \\
&= (x-2)^2(x+2)^2.
\end{aligned}$$

So the Jordan normal form contains either a single Jordan block $J_2(2)$ corresponding to eigenvalue 2 or two blocks $J_1(2)$ of size 1. Similar observations apply to the Jordan block(s) corresponding to the eigenvalue -2. To determine which occurs, we consider the minimum polynomial.

We now know the minimum polynomial of T has the form $m_T(x) = (x-2)^i(x+2)^j$ where $1 \leqslant i, j \leqslant 2$ by Corollary 4.19 and Theorem 4.20. Now

$$B - 2I = \begin{pmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 4 \\ 4 & 5 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \qquad \text{and} \qquad B + 2I = \begin{pmatrix} 4 & 1 & 0 & -3 \\ 0 & 4 & 0 & 4 \\ 4 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$(B-2I)(B+2I) = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

and

$$\begin{aligned}
(B-2I)^2(B+2I) &= \begin{pmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 4 \\ 4 & 5 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The first calculation shows $m_T(x)$ is not equal to the only possibility of degree 2, so we conclude from the second calculation that

$$m_T(x) = (x-2)^2(x+2).$$

Hence at least one Jordan block $J_2(2)$ of size 2 occurs in the Jordan normal form of T, while all Jordan blocks corresponding to the eigenvalue -2 have size 1.

We conclude the Jordan normal form of T is

$$\begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(-2) & 0 \\ 0 & 0 & J_1(-2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

□

Example 5.9 *Let $V = \mathbb{R}^4$ and let $T\colon V \to V$ be the linear transformation given by the matrix*

$$C = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Determine the Jordan normal form of T.

Solution:

$$\begin{aligned} c_T(x) &= \det \begin{pmatrix} x-3 & 0 & -1 & 1 \\ -1 & x-2 & -1 & 1 \\ 1 & 0 & x-1 & -1 \\ 0 & 0 & 0 & x-2 \end{pmatrix} \\ &= (x-2) \det \begin{pmatrix} x-3 & 0 & -1 \\ -1 & x-2 & -1 \\ 1 & 0 & x-1 \end{pmatrix} \\ &= (x-2)^2 \det \begin{pmatrix} x-3 & -1 \\ 1 & x-1 \end{pmatrix} \\ &= (x-2)^2 \big((x-3)(x-1) + 1 \big) \\ &= (x-2)^2 (x^2 - 4x + 3 + 1) \\ &= (x-2)^2 (x^2 - 4x + 4) \\ &= (x-2)^4. \end{aligned}$$

Now we calculate the minimum polynomial:

$$C - 2I = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$\begin{aligned} (C-2I)^2 &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $m_T(x) = (x-2)^2$. We now know the Jordan normal form for T includes at least one block $J_2(2)$ but we cannot tell whether the remaining blocks are a single block of size 2 or two blocks of size 1.

To actually determine which, we need to go beyond the characteristic and minimum polynomials, and consider the eigenspace E_2. We shall describe this in general and return to complete the solution of this example later. □

Consider a linear transformation $T\colon V \to V$ with Jordan normal form A. Each block $J_n(\lambda)$ occurring in A contributes one linearly independent eigenvector to a basis for the eigenspace E_λ (see Proposition 5.5(iii)). Thus the number of blocks in A corresponding to a particular eigenvalue λ will equal

$$\dim E_\lambda = n_\lambda,$$

the geometric multiplicity of λ. In summary:

Observation 5.10 *The geometric multiplicity of λ as an eigenvalue of T equals the number of Jordan blocks $J_n(\lambda)$ occurring in the Jordan normal form for T.*

Solution to Example 5.9 (cont.): Let us determine the eigenspace E_2 for our transformation T with matrix C. We solve $(T-2I)(\boldsymbol{v}) = \mathbf{0}$; that is,

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to a single equation

$$x + z - t = 0,$$

so

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

Hence $n_2 = \dim E_2 = 3$. It follows that the Jordan normal for T contains three Jordan blocks corresponding to the eigenvalue 2. Therefore the Jordan normal form of T is

$$\begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(2) & 0 \\ 0 & 0 & J_1(2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

This completes the solution. □

Our three observations are enough to determine the Jordan normal form for the linear transformations that will be encountered in this course. They are sufficient for small matrices, but will not solve the problem for all possibilities. For example, they do not distinguish between the 7×7 matrices

$$\begin{pmatrix} J_3(\lambda) & 0 & 0 \\ 0 & J_2(\lambda) & 0 \\ 0 & 0 & J_2(\lambda) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J_3(\lambda) & 0 & 0 \\ 0 & J_3(\lambda) & 0 \\ 0 & 0 & J_1(\lambda) \end{pmatrix},$$

which both have characteristic polynomial $(x-\lambda)^7$, minimum polynomial $(x-\lambda)^3$ and geometric multiplicity $n_\lambda = \dim E_\lambda = 3$. To deal with such possible Jordan normal forms one needs to generalise Observation 5.10 to consider the dimension of generalisations of eigenspaces:

$$\dim\ker(T-\lambda I)^2, \quad \dim\ker(T-\lambda I)^3, \quad \dots.$$

We leave the details to the interested and enthused student. By this point, the matrices will be sufficiently large that one should also make greater use of computer calculations to reduce the likelihood of hand-calculation errors.

We finish this section by returning to the final part of the general problem: finding a basis with respect to which a linear transformation is in Jordan normal form.

Example 5.11 *Let*

$$C = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(the matrix from Example 5.9). Find an invertible matrix P such that $P^{-1}CP$ is in Jordan normal form.

Solution: We have already established the Jordan normal form of the transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with matrix C is

$$A = \begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(2) & 0 \\ 0 & 0 & J_1(2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Our problem is equivalent (by Theorem 2.12) to finding a basis $\mathscr{B}$ for $\mathbb{R}^4$ with respect to which the matrix of T equals A. Thus $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ such that

$$T(\boldsymbol{v}_1) = 2\boldsymbol{v}_1, \qquad T(\boldsymbol{v}_2) = \boldsymbol{v}_1 + 2\boldsymbol{v}_2, \qquad T(\boldsymbol{v}_3) = 2\boldsymbol{v}_3, \qquad T(\boldsymbol{v}_4) = 2\boldsymbol{v}_4.$$

So we need to choose $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ to lie in the eigenspace E_2 (which we determined earlier).

On the face of it, the choice of $\boldsymbol{v}_2$ appears to be less straightforward: we require $(T-2I)(\boldsymbol{v}_2) = \boldsymbol{v}_1$, some non-zero vector in E_2 and this indicates we also probably do not have total freedom in the choice of $\boldsymbol{v}_1$. In Example 5.9, we calculated

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}.$$

Let us solve

$$(T-2I)(v) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix}.$$

We need to establish for which values of x, y and z this has a non-zero solution (and in the process we determine possibilities for both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$). The above matrix equation implies

$$\alpha + \gamma - \delta = x = y = -z$$

and

$$x + z = 0.$$

Any value of x will determine a possible solution, so let us choose $x = 1$. Then $y = 1$ and $z = -1$. Hence we shall take

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and then the equation $(T - 2I)(\boldsymbol{v}_2) = \boldsymbol{v}_1$ has non-zero solutions, namely

$$\boldsymbol{v}_2 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad \text{where} \quad \alpha + \gamma - \delta = 1.$$

There are many possible solutions, we shall take $\alpha = 1,\ \beta = \gamma = \delta = 0$ and hence

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

will be good enough.

To find $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$, we need two vectors from E_2 which together with $\boldsymbol{v}_1$ form a basis for E_2. We shall choose

$$\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Indeed, note that an arbitrary vector in E_2 can be expressed as

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \\ x + z \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + (y - x) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (x + z) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\ &= x\boldsymbol{v}_1 + (y - x)\boldsymbol{v}_3 + (x + z)\boldsymbol{v}_4, \end{aligned}$$

so $E_2 = \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_4)$.

We now have our required basis

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and the required change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[Exercise: Calculate $P^{-1}CP$ and verify it has the correct form.] □

Example 5.12 *Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by the matrix*

$$D = \begin{pmatrix} -3 & 2 & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Determine the Jordan normal form J of T and find an invertible matrix P such that $P^{-1}DP = J$.

Solution:

$$\begin{aligned} c_T(x) &= \det \begin{pmatrix} x+3 & -2 & -\frac{1}{2} & 2 \\ 0 & x & 0 & 0 \\ 0 & 3 & x+3 & 3 \\ 0 & 0 & 0 & x \end{pmatrix} \\ &= (x+3) \det \begin{pmatrix} x & 0 & 0 \\ 3 & x+3 & 3 \\ 0 & 0 & x \end{pmatrix} \\ &= x(x+3) \det \begin{pmatrix} x & 0 \\ 3 & x+3 \end{pmatrix} \\ &= x^2(x+3)^2. \end{aligned}$$

So the eigenvalues of T are 0 and -3. Then $m_T(x) = x^i(x+3)^j$ where $1 \leqslant i, j \leqslant 2$. Since

$$D + 3I = \begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

we calculate

$$D(D+3I) = \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

and

$$\begin{aligned} D(D+3I)^2 &= \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $m_T(x) = x(x+3)^2$. Therefore the Jordan normal form of T is

$$J = \begin{pmatrix} J_1(0) & 0 & 0 \\ 0 & J_1(0) & 0 \\ 0 & 0 & J_2(-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

We now seek a basis $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ for $\mathbb{R}^4$ with respect to which the matrix of T is J. Thus, we require $\boldsymbol{v}_1, \boldsymbol{v}_2 \in E_0,\ \ \boldsymbol{v}_3 \in E_{-3}$ and

$$T(\boldsymbol{v}_4) = \boldsymbol{v}_3 - 3\boldsymbol{v}_4.$$

We first solve $T(\boldsymbol{v}) = \mathbf{0}$:

$$\begin{pmatrix} -3 & 2 & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$\begin{aligned} -3x + 2y + \tfrac{1}{2}z - 2t &= 0 \\ -3y - 3z - 3t &= 0. \end{aligned}$$

Hence given arbitrary $z, t \in \mathbb{R}$, it follows that $y = -z - t$ and

$$\begin{aligned} x &= \tfrac{1}{3}(2y + \tfrac{1}{2}z - 2t) \\ &= \tfrac{1}{3}(-\tfrac{3}{2}z - 4t) \\ &= -\tfrac{1}{2}z - \tfrac{4}{3}t. \end{aligned}$$

So

$$E_0 = \left\{ \begin{pmatrix} -\frac{1}{2}z - \frac{4}{3}t \\ -z - t \\ z \\ t \end{pmatrix} \;\middle|\; z, t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix} \right).$$

Take

$$\boldsymbol{v}_1 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now solve $(T + 3I)(\boldsymbol{v}) = \mathbf{0}$:

$$\begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} z \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$\begin{aligned} 2y + \tfrac{1}{2}z - 2t &= 0 \\ 3y \phantom{+ \tfrac{1}{2}z - 2t} &= 0 \\ -3y \phantom{+ \tfrac{1}{2}z} - 3t &= 0 \\ 3t &= 0. \end{aligned}$$

Hence $y = t = 0$ and we deduce $z = 0$, while x may be arbitrary. Thus

$$E_{-3} = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

Take

$$\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We finally solve $T(\boldsymbol{v}_4) = \boldsymbol{v}_3 - 3\boldsymbol{v}_4$; that is, $(T + 3I)(\boldsymbol{v}_4) = \boldsymbol{v}_3$:

$$\begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} 2y + \tfrac{1}{2}z - 2t &= 1 \\ 3y \qquad\qquad &= 0 \\ -3y \qquad - 3t &= 0 \\ 3t &= 0, \end{aligned}$$

so $y = t = 0$ and then $\frac{1}{2}z = 1$, which forces $z = 2$, while x may be arbitrary. Thus

$$\boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

is one solution. Thus

$$\mathscr{B} = \left\{ \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

and our change of basis matrix is

$$P = \begin{pmatrix} -\frac{1}{2} & -\frac{4}{3} & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

□

This last example illustrates some general principles. When seeking the invertible matrix P such that $P^{-1}AP$ is in Jordan normal form, we seek particular vectors to form a basis. These basis vectors can be found by solving appropriate systems of linear equations (though sometimes care is needed to find the correct system to solve as was illustrated in Example 5.11).

Example 5A *Let $V = \mathbb{R}^5$ and let $T \colon V \to V$ be the linear transformation given by the matrix*

$$E = \begin{pmatrix} 1 & 0 & -1 & 0 & -8 \\ 0 & 1 & 4 & 0 & 29 \\ -1 & 0 & 1 & 1 & 5 \\ 0 & 0 & -1 & 1 & -11 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Determine a Jordan normal form J of T and find an invertible matrix P such that $P^{-1}EP = J$.

Solution: We first determine the characteristic polynomial of T:

$$\begin{aligned}
c_T(x) &= \det \begin{pmatrix} x-1 & 0 & 1 & 0 & 8 \\ 0 & x-1 & -4 & 0 & -29 \\ 1 & 0 & x-1 & -1 & -5 \\ 0 & 0 & 1 & x-1 & 11 \\ 0 & 0 & 0 & 0 & x+2 \end{pmatrix} \\
&= (x+2) \det \begin{pmatrix} x-1 & 0 & 1 & 0 \\ 0 & x-1 & -4 & 0 \\ 1 & 0 & x-1 & -1 \\ 0 & 0 & 1 & x-1 \end{pmatrix} \\
&= (x-1)(x+2) \det \begin{pmatrix} x-1 & 1 & 0 \\ 1 & x-1 & -1 \\ 0 & 1 & x-1 \end{pmatrix} \\
&= (x-1)(x+2) \left((x-1) \det \begin{pmatrix} x-1 & -1 \\ 1 & x-1 \end{pmatrix} - \det \begin{pmatrix} 1 & -1 \\ 0 & x-1 \end{pmatrix} \right) \\
&= (x-1)(x+2)\Big((x-1)\big((x-1)^2+1\big) - (x-1) \Big) \\
&= (x-1)^2(x+2)\big((x-1)^2+1-1\big) \\
&= (x-1)^4(x+2).
\end{aligned}$$

We now know that the Jordan normal form for T contains a single Jordan block $J_1(-2)$ corresponding to eigenvalue -2 and some number of Jordan blocks $J_n(1)$ corresponding to eigenvalue 1. The sum of the sizes of these latter blocks equals 4.

Let us now determine the minimum polynomial of T. We know $m_T(x) = (x-1)^i(x+2)$ where $1 \leqslant i \leqslant 4$ by Corollary 4.19 and Theorem 4.20. Now

$$E - I = \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad E + 2I = \begin{pmatrix} 3 & 0 & -1 & 0 & -8 \\ 0 & 3 & 4 & 0 & 29 \\ -1 & 0 & 3 & 1 & 5 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$\begin{aligned}
(E-I)(E+2I) &= \begin{pmatrix} 1 & 0 & -3 & -1 & -5 \\ -4 & 0 & 12 & 4 & 20 \\ -3 & 0 & 0 & 3 & -3 \\ 1 & 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \neq 0, \\
(E-I)^2(E+2I) &= \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & -1 & -5 \\ -4 & 0 & 12 & 4 & 20 \\ -3 & 0 & 0 & 3 & -3 \\ 1 & 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 0 & -3 & 3 \\ -12 & 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \neq 0
\end{aligned}$$

and

$$(E-I)^3(E+2I) = \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & -3 & 3 \\ -12 & 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $m_T(x) = (x-1)^3(x+2)$. As a consequence, the Jordan normal form of T must contain at least one Jordan block $J_3(1)$ of size 3. Since the sizes of the Jordan blocks associated to the eigenvalue 1 has sum equal to 4 (from earlier), there remains a single Jordan block $J_1(1)$ of size 1.

Our conclusion is a Jordan normal form of T is

$$J = \begin{pmatrix} J_3(1) & 0 & 0 \\ 0 & J_1(1) & 0 \\ 0 & 0 & J_1(-2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

We now want to find a basis $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ for $\mathbb{R}^5$ such that $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = J$. In particular, $\boldsymbol{v}_1$ and $\boldsymbol{v}_4$ are required to be eigenvectors with eigenvalue 1. Let us first find the eigenspace E_1 by solving $(T-I)(\boldsymbol{v}) = \boldsymbol{0}$; that is,

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The fifth row yields $-3u = 0$; that is, $u = 0$. It follows from the first row that $-z - 8u = 0$ and hence $z = 0$. The only row yielding further information is the third which says $-x + t + 5u = 0$ and so $x = t$. Hence

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ x \\ 0 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}.$$

From this we can read off a basis for the eigenspace E_1, but this does not tell us which vector to take as $\boldsymbol{v}_1$. We need $\boldsymbol{v}_1$ to be a suitable choice of eigenvector so that $T(\boldsymbol{v}_2) = \boldsymbol{v}_1 + \boldsymbol{v}_2$, that is, $(T-I)(\boldsymbol{v}_2) = \boldsymbol{v}_1$, is possible. We solve for $(T-I)(\boldsymbol{v}) = \boldsymbol{w}$ where $\boldsymbol{w}$ is a typical vector in E_1. So consider

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \alpha \\ 0 \end{pmatrix}$$

for some non-zero scalars α and β. Thus $-3u = 0$ and so $u = 0$. We then obtain three equations

$$-z = \alpha, \qquad 4z = \beta, \qquad -x + t = 0. \tag{5.1}$$

Thus to have a solution it must be the case that $-4\alpha = \beta$. This tells us what to take as $\boldsymbol{v}_1$: we want a vector of the form

$$\begin{pmatrix} \alpha \\ -4\alpha \\ 0 \\ \alpha \\ 0 \end{pmatrix} \qquad \text{where } \alpha \in \mathbb{R}, \text{ but } \alpha \neq 0.$$

So take $\alpha = 1$ and

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then Equations (5.1) tell us that vector $\boldsymbol{v}_2$ is given by $z = -1$ and $x = t$ but give no further restriction on x and y. These last two scalars can be arbitrary, so we shall take $x = y = 0$ (mainly for convenience):

$$\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\boldsymbol{v}_3$ is required to satisfy $T(\boldsymbol{v}_3) = \boldsymbol{v}_2 + \boldsymbol{v}_3$, so to find $\boldsymbol{v}_3$ we solve $(T - I)(\boldsymbol{v}) = \boldsymbol{v}_2$:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $u = 0$, $z = 0$ and $-x + t = -1$. Any other choices are arbitrary, so we shall take $x = 1$, $y = 0$ and then $t = 0$. So we take

$$\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\boldsymbol{v}_3$, we note that $\boldsymbol{v}_1$, $\boldsymbol{v}_4$ should be linearly independent vectors (as they form part of a basis) in the eigenspace

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ x \\ 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}.$$

Having already chosen the vector $\boldsymbol{v}_1$, we must choose $\boldsymbol{v}_4$ not to be a scalar multiple of $\boldsymbol{v}_1$. One possible choice, and the one we shall take, is

$$\boldsymbol{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(i.e., take $x = 1$, $y = 0$).

Finally, we require $\boldsymbol{v}_5$ to be an eigenvector for T with eigenvalue -2, so we solve $T(\boldsymbol{v}) = -2\boldsymbol{v}$ or, equivalently, $(T + 2I)(\boldsymbol{v}) = \mathbf{0}$:

$$\begin{pmatrix} 3 & 0 & -1 & 0 & -8 \\ 0 & 3 & 4 & 0 & 29 \\ -1 & 0 & 3 & 1 & 5 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We apply row operations to solve this system of equations:

$$\left(\begin{array}{ccccc|c} 3 & 0 & -1 & 0 & -8 & 0 \\ 0 & 3 & 4 & 0 & 29 & 0 \\ -1 & 0 & 3 & 1 & 5 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 8 & 3 & 7 & 0 \\ 0 & 3 & 4 & 0 & 29 & 0 \\ -1 & 0 & 3 & 1 & 5 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad r_1 \mapsto r_1 + 3r_3$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 27 & -81 & 0 \\ 0 & 3 & 0 & 12 & -15 & 0 \\ -1 & 0 & 0 & 10 & -28 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto r_1 + 8r_4 \\ r_2 \mapsto r_2 + 4r_4 \\ r_3 \mapsto r_3 + 3r_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 4 & -5 & 0 \\ -1 & 0 & 0 & 10 & -28 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto \frac{1}{27} r_1 \\ r_2 \mapsto \frac{1}{3} r_2 \end{array}$$

Hence

$$\begin{aligned} t - 3u &= 0 \\ y + 4t - 5u &= 0 \\ -x + 10t - 28u &= 0 \\ -z + 3t - 11u &= 0. \end{aligned}$$

Take $u = 1$ (it can be non-zero, but otherwise arbitrary, when producing the eigenvector $\boldsymbol{v}_5$). Then

$$\begin{aligned} t &= 3u = 3 \\ y &= 5u - 4t = -7 \\ x &= 10t - 28u = 2 \\ z &= 3t - 11u = -2. \end{aligned}$$

So we take

$$\boldsymbol{v}_5 = \begin{pmatrix} 2 \\ -7 \\ -2 \\ 3 \\ 1 \end{pmatrix}.$$

With the above choices, the matrix of T with respect to the basis $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ is then our Jordan normal form J. The change of basis matrix P such that $P^{-1}EP = J$ is found by writing each $\boldsymbol{v}_j$ in terms of the standard basis and placing the coefficients in the jth column of P. Thus

$$P = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -4 & 0 & 0 & 0 & -7 \\ 0 & -1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Chapter 6

Inner product spaces

We now introduce a new topic, namely vector spaces endowed with an inner product. This is a generalisation of the dot (or scalar) product of vectors in $\mathbb{R}^3$ and are extremely important. In the final chapter of the notes, we shall describe links between inner product spaces and diagonalisation of (suitable) linear transformations.

Throughout this section (and the rest of the course), our base field F will be either $\mathbb{R}$ or $\mathbb{C}$. Recall that if $z = x + iy \in \mathbb{C}$, the *complex conjugate* of z is given by

$$\bar{z} = x - iy.$$

To save space and time, we shall use the complex conjugate even when $F = \mathbb{R}$. Thus, when $F = \mathbb{R}$ and $\bar{\alpha}$ appears for a scalar α in $\mathbb{R}$, it will mean $\bar{\alpha} = \alpha$.

Definition 6.1 Let $F = \mathbb{R}$ or $\mathbb{C}$. An *inner product space* is a vector space V over F together with an *inner product*

$$\begin{aligned} V \times V &\to F \\ (v, w) &\mapsto \langle v, w \rangle \end{aligned}$$

such that

(i) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$,

(ii) $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $v, w \in V$ and $\alpha \in F$,

(iii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$,

(iv) $\langle v, v \rangle$ is a real number satisfying $\langle v, v \rangle \geqslant 0$ for all $v \in V$,

(v) $\langle v, v \rangle = 0$ if and only if $v = \mathbf{0}$.

Thus, in the case **when $\boldsymbol{F} = \mathbb{R}$**, our inner product is symmetric in the sense that Condition (iii) then becomes

$$\langle v, w \rangle = \langle w, v \rangle \qquad \text{for all } v, w \in V.$$

Example 6.2 (i) The vector space $\mathbb{R}^n$ of column vectors of real numbers is an inner product space with respect to the usual *dot product*:

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i.$$

Note that if $\boldsymbol{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} x_i^2$$

and from this Condition (iv) follows immediately.

(ii) We can endow $\mathbb{C}^n$ with an inner product by introducing the complex conjugate:

$$\left\langle \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} z_i \bar{w}_i.$$

Note that if $\boldsymbol{v} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$, then

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} z_i \bar{z}_i = \sum_{i=1}^{n} |z_i|^2.$$

(iii) If $a < b$, the set $C[a, b]$ of continuous functions $f \colon [a, b] \to \mathbb{R}$ is a real vector space when we define

$$(f + g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha \cdot f(x)$$

for $f, g \in C[a, b]$ and $\alpha \in \mathbb{R}$. In fact, $C[a, b]$ is an inner product space when we define

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, \mathrm{d}x.$$

Since $f(x)^2 \geqslant 0$ for all x, it follows that

$$\langle f, f \rangle = \int_a^b f(x)^2 \, \mathrm{d}x \geqslant 0.$$

(iv) The space $\mathcal{P}_n$ of real polynomials of degree at most n is a real vector space of dimension $n + 1$. It becomes an inner product space by inheriting the inner product from above, for example:

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, \mathrm{d}x$$

for real polynomials $f(x), g(x) \in \mathcal{P}_n$.

We can also generalise these last two examples to complex-valued functions. For example, the complex vector space of polynomials

$$f(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x + \alpha_0$$

where $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ becomes an inner product space when we define

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$$

where

$$\overline{f(x)} = \bar{\alpha}_n x^n + \bar{\alpha}_{n-1} x^{n-1} + \dots + \bar{\alpha}_1 x + \bar{\alpha}_0.$$

Definition 6.3 Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. The *norm* is the function $\| \cdot \| \colon V \to \mathbb{R}$ defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

(This makes sense since $\langle v, v \rangle \geqslant 0$ for all $v \in V$.)

Lemma 6.4 *Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Then*

(i) $\langle v, \alpha w \rangle = \bar{\alpha} \langle v, w \rangle$ *for all $v, w \in V$ and $\alpha \in F$;*

(ii) $\|\alpha v\| = |\alpha| \cdot \|v\|$ *for all $v \in V$ and $\alpha \in F$;*

(iii) $\|v\| > 0$ *whenever $v \neq \mathbf{0}$.*

Note that it is the modulus $|\alpha|$ of the (real or complex) number α that appears in part (ii) here.

Proof: (i)

$$\langle v, \alpha w \rangle = \overline{\langle \alpha w, v \rangle} = \overline{\alpha \langle w, v \rangle} = \bar{\alpha}\overline{\langle w, v \rangle} = \bar{\alpha} \langle v, w \rangle.$$

(ii)

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v \rangle = \alpha \langle v, \alpha v \rangle = \alpha \bar{\alpha} \langle v, v \rangle = |\alpha|^2 \, \|v\|^2$$

and taking square roots gives the result.

(iii) $\langle v, v \rangle > 0$ whenever $v \neq \mathbf{0}$. □

Theorem 6.5 (Cauchy–Schwarz Inequality) *Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Then*

$$|\langle u, v \rangle| \leqslant \|u\| \cdot \|v\|$$

for all $u, v \in V$.

Proof: If $v = \mathbf{0}$, then we see

$$\langle u, v \rangle = \langle u, \mathbf{0} \rangle = \langle u, 0 \cdot \mathbf{0} \rangle = 0 \langle u, \mathbf{0} \rangle = 0.$$

Hence

$$|\langle u, v \rangle| = 0 = \|u\| \cdot \|v\|$$

as $\|v\| = 0$.

In the remainder of the proof we assume $v \neq \mathbf{0}$. Let α be a scalar, put $w = u + \alpha v$ and expand $\langle w, w \rangle$:

$$\begin{aligned} 0 \leqslant \langle w, w \rangle &= \langle u + \alpha v, u + \alpha v \rangle \\ &= \langle u, u \rangle + \alpha \langle v, u \rangle + \bar{\alpha} \langle u, v \rangle + \alpha \bar{\alpha} \langle v, v \rangle \\ &= \|u\|^2 + \alpha \overline{\langle u, v \rangle} + \bar{\alpha} \langle u, v \rangle + |\alpha|^2 \|v\|^2. \end{aligned}$$

Now take $\alpha = -\langle u, v \rangle / \|v\|^2$. We deduce

$$\begin{aligned} 0 &\leqslant \|u\|^2 - \frac{\langle u, v \rangle \cdot \overline{\langle u, v \rangle}}{\|v\|^2} - \frac{\overline{\langle u, v \rangle} \langle u, v \rangle}{\|v\|^2} + \frac{|\langle u, v \rangle|^2}{\|v\|^4} \|v\|^2 \\ &= \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2}, \end{aligned}$$

so

$$|\langle u, v \rangle|^2 \leqslant \|u\|^2 \|v\|^2$$

and taking square roots gives the result. □

Corollary 6.6 (Triangle Inequality) *Let V be an inner product space. Then*

$$\|u + v\| \leqslant \|u\| + \|v\|$$

for all $u, v \in V$.

PROOF:

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2 \\ &= \|u\|^2 + 2 \operatorname{Re} \langle u, v \rangle + \|v\|^2 \\ &\leqslant \|u\|^2 + 2 |\langle u, v \rangle| + \|v\|^2 \\ &\leqslant \|u\|^2 + 2 \|u\| \cdot \|v\| + \|v\|^2 && \text{(by Cauchy–Schwarz)} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

and taking square roots gives the result. □

The triangle inequality is a fundamental observation that tells us we can use the norm to measure distance on an inner product space in the same way that modulus $|x|$ is used to measure distance on $\mathbb{R}$ or $\mathbb{C}$. We can then perform analysis and speak of continuity and convergence. This topic is addressed in greater detail in the study of Functional Analysis.

Orthogonality and orthonormal bases

Definition 6.7 Let V be an inner product space.

(i) Two vectors v and w are said to be *orthogonal* if $\langle v, w \rangle = 0$.

(ii) A set $\mathscr{A}$ of vectors is *orthogonal* if every pair of distinct vectors within it are orthogonal.

(iii) A set $\mathscr{A}$ of vectors is *orthonormal* if it is orthogonal and every vector in $\mathscr{A}$ has unit norm.

Thus a set $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ is orthonormal if

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

for all $i, j \in \{1, 2, \dots, k\}$. (Here, again, we use the Kronecker delta as an abbreviation for the formula on the right-hand side.)

An *orthonormal basis* for an inner product space V is a basis which is itself an orthonormal set.

Example 6.8 (i) The standard basis $\mathscr{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

(ii) Consider the inner product space $C[-\pi, \pi]$, consisting of all continuous functions $f \colon [-\pi, \pi] \to \mathbb{R}$, with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, \mathrm{d}x.$$

Define

$$\begin{aligned} e_0(x) &= \frac{1}{\sqrt{2\pi}} \\ e_n(x) &= \frac{1}{\sqrt{\pi}} \cos nx \\ f_n(x) &= \frac{1}{\sqrt{\pi}} \sin nx \end{aligned}$$

for $n = 1, 2, \dots$. These functions (without the scaling) were studied in *MT2507*. In that module, the following facts were established:

$$\begin{aligned} \langle e_m, e_n \rangle &= 0 && \text{if } m \neq n, \\ \langle f_m, f_n \rangle &= 0 && \text{if } m \neq n, \\ \langle e_m, f_n \rangle &= 0 && \text{for all } m, n \end{aligned}$$

and

$$\|e_n\| = \|f_n\| = 1 \qquad \text{for all } n.$$

(The reason for the scaling factors in the definitions of the e_n and f_n is to achieve unit norm for each function.) The topic of Fourier series relates to expressing functions as linear combinations of the orthonormal set

$$\{\, e_0, e_n, f_n \mid n = 1, 2, 3, \dots \,\}.$$

As an aside, we mention that this set is *not* a basis for $C[-\pi, \pi]$. Instead, it is what is called a *complete orthonormal set*, which means that every function in $C[-\pi, \pi]$ can be expressed as a convergent series

$$f = \sum_{n=0}^{\infty} \alpha_n e_n + \sum_{n=1}^{\infty} \beta_n f_n$$

where each $\alpha_n, \beta_n \in \mathbb{R}$. To explain this fully one needs to talk about convergence of series in (complete) inner product spaces and we refer to *MT4515 Functional Analysis* for more details.

Theorem 6.9 *An orthogonal set of non-zero vectors is linearly independent.*

Proof: Let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be an orthogonal set of non-zero vectors. Suppose that

$$\sum_{i=1}^{k} \alpha_i v_i = \mathbf{0}.$$

Then, by linearity of the inner product in the first entry, for $j = 1, 2, \dots, k$ we determine that

$$0 = \left\langle \sum_{i=1}^{k} \alpha_i v_i, v_j \right\rangle = \sum_{i=1}^{k} \alpha_i \langle v_i, v_j \rangle = \alpha_j \|v_j\|^2,$$

since by assumption $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Now $v_j \neq \mathbf{0}$, so $\|v_j\| \neq 0$. Hence we conclude

$$\alpha_j = 0 \qquad \text{for all } j.$$

Thus $\mathscr{A}$ is linearly independent. □

One might ask how one constructs an orthonormal basis in an inner product space? The following algorithm provides the answer to this question:

Theorem 6.10 (Gram–Schmidt Process) *Suppose that V is a finite-dimensional inner product space with basis $\{v_1, v_2, \dots, v_n\}$. Consider the following procedure:*

Step 1: *Define* $e_1 = \dfrac{1}{\|v_1\|} v_1$.

Step k: *Suppose $\{e_1, e_2, \dots, e_{k-1}\}$ has been constructed. Define*

$$w_k = v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i$$

and

$$e_k = \frac{1}{\|w_k\|} w_k.$$

Then this defines an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for V.

Proof: We claim that, for $k = 1, 2, \dots, n$, the set $\{e_1, e_2, \dots, e_k\}$ is defined, orthonormal, and contained in $\mathrm{Span}(v_1, v_2, \dots, v_k)$.

Step 1: v_1 is a non-zero vector, so $\|v_1\| \neq 0$ and hence $e_1 = \frac{1}{\|v_1\|} v_1$ is defined. Now

$$\|e_1\| = \left\| \frac{1}{\|v_1\|} v_1 \right\| = \frac{1}{\|v_1\|} \cdot \|v_1\| = 1.$$

Hence $\{e_1\}$ is an orthonormal set (there are no orthogonality conditions to check) and by definition $e_1 \in \mathrm{Span}(v_1)$.

Step *k*: Suppose, as an inductive hypothesis, that we have shown $\{e_1, e_2, \dots, e_{k-1}\}$ is an orthonormal set contained in $\mathrm{Span}(v_1, v_2, \dots, v_{k-1})$. Consider

$$w_k = v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle.$$

We claim that $w_k \neq \mathbf{0}$. Indeed, if $w_k = \mathbf{0}$, then

$$\begin{aligned} v_k = \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i &\in \mathrm{Span}(e_1, \dots, e_{k-1}) \\ &\subseteq \mathrm{Span}(v_1, \dots, v_{k-1}). \end{aligned}$$

But this contradicts $\{v_1, v_2, \dots, v_n\}$ being linearly independent. Thus $w_k \neq \mathbf{0}$ and hence $e_k = \frac{1}{\|w_k\|} w_k$ is defined.

By construction $\|e_k\| = 1$ and

$$\begin{aligned} e_k &= \frac{1}{\|w_k\|} \left(v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i \right) \\ &\in \mathrm{Span}(e_1, \dots, e_{k-1}, v_k) \\ &\subseteq \mathrm{Span}(v_1, \dots, v_{k-1}, v_k). \end{aligned}$$

It remains to check that e_k is orthogonal to e_j for $j = 1, 2, \dots, k-1$. We calculate

$$\begin{aligned} \langle w_k, e_j \rangle &= \left\langle v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i, e_j \right\rangle \\ &= \langle v_k, e_j \rangle - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle \langle e_i, e_j \rangle \\ &= \langle v_k, e_j \rangle - \langle v_k, e_j \rangle \|e_j\|^2 && \text{(by inductive hypothesis)} \\ &= \langle v_k, e_j \rangle - \langle v_k, e_j \rangle = 0. \end{aligned}$$

Hence

$$\langle e_k, e_j \rangle = \left\langle \frac{1}{\|w_k\|} w_k, e_j \right\rangle = \frac{1}{\|w_k\|} \langle w_k, e_j \rangle = 0$$

for $j = 1, 2, \dots, k-1$.

This completes the induction. We conclude that, at the final stage, $\{e_1, e_2, \dots, e_n\}$ is an orthonormal set. Theorem 6.9 tells us this set is linearly independent and hence a basis for V (since $\dim V = n$). □

Example 6.11 *Consider $\mathbb{R}^3$ with the usual inner product. Find an orthonormal basis for the subspace U spanned by the vectors*

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

SOLUTION: We apply the Gram–Schmidt Process to $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

$$\|\boldsymbol{v}_1\|^2 = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 1^2 + (-1)^2 = 2.$$

Take

$$\boldsymbol{e}_1 = \frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Now

$$\langle \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle = \left\langle \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = \frac{1}{\sqrt{2}}(2-1) = \frac{1}{\sqrt{2}}.$$

Put

$$\begin{aligned} \boldsymbol{w}_2 &= \boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 \\ &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \\ 3/2 \end{pmatrix}. \end{aligned}$$

So

$$\|\boldsymbol{w}_2\|^2 = (3/2)^2 + 3^2 + (3/2)^2 = \frac{27}{2}$$

and

$$\|\boldsymbol{w}_2\| = \frac{3\sqrt{3}}{\sqrt{2}}.$$

Take

$$\boldsymbol{e}_2 = \frac{1}{\|\boldsymbol{w}_2\|}\boldsymbol{w}_2 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Thus

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is an orthonormal basis for U. □

Example 6.12 (Laguerre polynomials) We can define an inner product on the space $\mathcal{P}$ of real polynomials $f(x)$ by

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)\mathrm{e}^{-x}\,\mathrm{d}x.$$

The *Laguerre polynomials* form the orthonormal basis for $\mathcal{P}$ that is produced when we apply the Gram–Schmidt process to the standard basis

$$\{1, x, x^2, x^3, \dots\}$$

of monomials.

Determine the first three Laguerre polynomials.

SOLUTION: We apply the Gram–Schmidt process to the basis $\{1, x, x^2\}$ for the inner product space $\mathcal{P}_2$, of polynomials of degree at most 2, with inner product as above. We shall make use of the fact (determined by induction and integration by parts) that

$$\int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x = n!$$

Define $f_i(x) = x^i$ for $i = 0, 1, 2$. Then

$$\|f_0\|^2 = \int_0^\infty f_0(x)^2 \mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1,$$

so

$$L_0(x) = \frac{1}{\|f_0\|} f_0(x) = 1.$$

We now calculate L_1. First

$$\langle f_1, L_0 \rangle = \int_0^\infty f_1(x) L_0(x) \mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x = 1.$$

The Gram-Schmidt process says we first define

$$w_1(x) = f_1(x) - \langle f_1, L_0 \rangle L_0(x) = x - 1.$$

Now

$$\begin{aligned}
\|w_1\|^2 &= \int_0^\infty w_1(x)^2 \mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty (x^2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} + \mathrm{e}^{-x})\,\mathrm{d}x \\
&= 2 - 2 + 1 = 1.
\end{aligned}$$

Hence

$$L_1(x) = \frac{1}{\|w_1\|} w_1(x) = x - 1.$$

In the next step of the Gram–Schmidt process, we calculate

$$\langle f_2, L_0 \rangle = \int_0^\infty x^2 \mathrm{e}^{-x}\,\mathrm{d}x = 2$$

and

$$\begin{aligned}
\langle f_2, L_1 \rangle &= \int_0^\infty x^2(x-1)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty (x^3\mathrm{e}^{-x} - x^2\mathrm{e}^{-x})\,\mathrm{d}x \\
&= 3! - 2! = 6 - 2 = 4.
\end{aligned}$$

So we define

$$\begin{aligned}
w_2(x) &= f_2(x) - \langle f_2, L_0 \rangle L_0(x) - \langle f_2, L_1 \rangle L_1(x) \\
&= x^2 - 4(x-1) - 2 \\
&= x^2 - 4x + 2.
\end{aligned}$$

Now

$$\begin{aligned}
\|w_2\|^2 &= \int_0^\infty w_2(x)^2 \mathrm{e}^{-x} \, \mathrm{d}x \\
&= \int_0^\infty (x^4 - 8x^3 + 20x^2 - 16x + 4)\mathrm{e}^{-x} \, \mathrm{d}x \\
&= 4! - 8 \cdot 3! + 20 \cdot 2! - 16 + 4 \\
&= 4.
\end{aligned}$$

Hence we take

$$L_2(x) = \frac{1}{\|w_2\|} w_2(x) = \tfrac{1}{2}(x^2 - 4x + 2).$$

Similar calculations can be performed to determine L_3, L_4, ..., but they become increasingly more complicated (and consequently less suitable for presenting on a whiteboard!). □

Example 6.13 Define an inner product on the space $\mathcal{P}$ of real polynomials by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, \mathrm{d}x.$$

Applying the Gram–Schmidt process to the monomials $\{1, x, x^2, x^3, \dots\}$ produces an orthonormal basis (with respect to this inner product). The polynomials produced are scalar multiples of the *Legendre polynomials*:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
&\vdots
\end{aligned}$$

The set $\{ P_n(x) \mid n = 0, 1, 2, \dots \}$ of Legendre polynomials is *orthogonal*, but *not* orthonormal. This is the reason why the Gram–Schmidt process only produces a scalar multiple of them. The scalars appearing are determined by the norms of the P_n with respect to this inner product.

For example,

$$\|P_0\|^2 = \int_{-1}^{1} P_0(x)^2 \, \mathrm{d}x = \int_{-1}^{1} \mathrm{d}x = 2,$$

so the polynomial of unit norm produced will be $\frac{1}{\sqrt{2}} P_0(x)$. Similar calculations (of increasing length) can be performed for the other polynomials.

The *Hermite polynomials* form an orthogonal set in the space $\mathcal{P}$ when we endow it with the following inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)\mathrm{e}^{-x^2/2} \, \mathrm{d}x.$$

Again the orthonomal basis produced by applying the Gram–Schmidt process to the monomials are scalar multiples of the Hermite polynomials.

The three families of polynomials we have just described (Laguerre, Legendre and Hemite polynomials) arise in multiple places, particularly applied mathematics and physics. In particular, they occur in the context of particular differential equations. As one example, the Laguerre polynomials are relevant to quantum mechanics and the Schrödinger equation. More details is beyond what can be covered in this lecture course, but their presence shows the links from inner product spaces to other branches of mathematics and science.

Orthogonal complements

Definition 6.14 Let V be an inner product space. If U is a subspace of V, the *orthogonal complement* to U is

$$U^{\perp} = \{\, v \in V \mid \langle v, u \rangle = 0 \text{ for all } u \in U \,\}.$$

Thus $U^{\perp}$ consists of those vectors which are orthogonal to every single vector in U.

Lemma 6.15 *Let V be an inner product space and U be a subspace of V. Then*

(i) *$U^{\perp}$ is a subspace of V, and*

(ii) $U \cap U^{\perp} = \{\mathbf{0}\}$.

PROOF: (i) First note, since the inner product is linear in the first entry, that $\langle \mathbf{0}, u \rangle = 0$ for all $u \in U$, so $\mathbf{0} \in U^{\perp}$. Now let $v, w \in U^{\perp}$ and $\alpha \in F$. Then

$$\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = 0 + 0 = 0$$

and

$$\langle \alpha v, u \rangle = \alpha \langle v, u \rangle = \alpha \cdot 0 = 0$$

for all $u \in U$. So we deduce $v + w \in U^{\perp}$ and $\alpha v \in U^{\perp}$. This shows that $U^{\perp}$ is a subspace.

(ii) Let $u \in U \cap U^{\perp}$. Then

$$\|u\|^2 = \langle u, u \rangle = 0$$

(since the element u is, in particular, orthogonal to itself). Hence $u = \mathbf{0}$. □

Theorem 6.16 *Let V be a finite-dimensional inner product space and U be a subspace of V. Then*

$$V = U \oplus U^{\perp}.$$

PROOF: We already know that $U \cap U^{\perp} = \{\mathbf{0}\}$, so it remains to show $V = U + U^{\perp}$.

Let $\{v_1, v_2, \dots, v_k\}$ be a basis for U. Extend it to a basis

$$\mathscr{B} = \{v_1, v_2, \dots, v_k, w_{k+1}, \dots, w_n\}$$

for V. Now apply the Gram–Schmidt process to $\mathscr{B}$ and hence produce an orthonormal basis $\mathscr{E} = \{e_1, e_2, \dots, e_n\}$ for V. By construction,

$$\{e_1, e_2, \dots, e_k\} \subseteq \operatorname{Span}(v_1, v_2, \dots, v_k) = U$$

and, since it is an orthonormal set, $\{e_1, e_2, \dots, e_k\}$ is a linearly independent set of size $k = \dim U$. Therefore $\{e_1, e_2, \dots, e_k\}$ is a basis for U.

Hence any vector $u \in U$ can be uniquely written as $u = \sum_{i=1}^{k} \alpha_i e_i$. Then for all such u

$$\langle u, e_j \rangle = \left\langle \sum_{i=1}^{k} \alpha_i e_i, e_j \right\rangle = \sum_{i=1}^{k} \alpha_i \langle e_i, e_j \rangle = 0$$

for $j = k+1, k+2, \dots, n$; that is,

$$e_{k+1}, e_{k+2}, \dots, e_n \in U^{\perp}.$$

Now if $v \in V$, we can write

$$v = \beta_1 e_1 + \cdots + \beta_k e_k + \beta_{k+1} e_{k+1} + \cdots + \beta_n e_n$$

for some scalars $\beta_1, \beta_2, \dots, \beta_n$ and

$$\beta_1 e_1 + \cdots + \beta_k e_k \in U \qquad \text{and} \qquad \beta_{k+1} e_{k+1} + \cdots + \beta_n e_n \in U^{\perp}.$$

This shows that every vector in V is the sum of a vector in U and one in $U^{\perp}$, so

$$V = U + U^{\perp} = U \oplus U^{\perp},$$

as required to complete the proof. □

Once we have a direct sum, we can consider an associated projection map. In particular, we have the projection $P_U \colon V \to V$ onto U associated to the decomposition $V = U \oplus U^{\perp}$. This is given by

$$P_U(v) = u$$

where $v = u + w$ is the unique decomposition of v with $u \in U$ and $w \in U^{\perp}$.

Theorem 6.17 *Let V be a finite-dimensional inner product space and U be a subspace of V. Let $P_U \colon V \to V$ be the projection map onto U associated to the direct sum decomposition $V = U \oplus U^{\perp}$. If $v \in V$, then $P_U(v)$ is the vector in U that is closest to v.*

By "closest" here, we are referring to distance as determined by the norm defined in Definition 6.3.

PROOF: Recall that the norm $\|\cdot\|$ determines the distance between two vectors, specifically $\|v - u\|$ is the distance from v to u. Write $v = u_0 + w_0$ where $u_0 \in U$ and $w_0 \in U^{\perp}$, so that $P_U(v) = u_0$. Then if u is any vector in U,

$$\begin{aligned}
\|v - u\|^2 &= \|v - u_0 + (u_0 - u)\|^2 \\
&= \|w_0 + (u_0 - u)\|^2 \\
&= \langle w_0 + (u_0 - u), w_0 + (u_0 - u) \rangle \\
&= \langle w_0, w_0 \rangle + \langle w_0, u_0 - u \rangle + \langle u_0 - u, w_0 \rangle + \langle u_0 - u, u_0 - u \rangle \\
&= \|w_0\|^2 + \|u_0 - u\|^2 \qquad \text{(since w_0 is orthogonal to $u_0 - u \in U$)} \\
&\geqslant \|w_0\|^2 \qquad \text{(since $\|u_0 - u\| \geqslant 0$)} \\
&= \|v - u_0\|^2 \\
&= \|v - P_U(v)\|^2.
\end{aligned}$$

Hence

$$\|v - u\| \geqslant \|v - P_U(v)\| \qquad \text{for all } u \in U.$$

This proves the theorem: $P_U(v)$ is closer to v than any other vector in U. □

Example 6.18 *Find the distance from the vector* $\boldsymbol{w}_0 = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$ *in* $\mathbb{R}^3$ *to the subspace*

$$U = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right).$$

Solution: We need to find $U^\perp$, which must be a 1-dimensional subspace since $\mathbb{R}^3 = U \oplus U^\perp$. We solve the condition $\langle \boldsymbol{v}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in U$:

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x + y + z$$

and

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = y - 2z.$$

Hence

$$x + y + z = y - 2z = 0.$$

Given arbitrary z, we take $y = 2z$ and $x = -y - z = -3z$. Therefore

$$U^\perp = \left\{ \begin{pmatrix} -3z \\ 2z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \right).$$

The closest vector in U to $\boldsymbol{w}_0$ is $P_U(\boldsymbol{w}_0)$ where $P_U\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the projection onto U associated to $\mathbb{R}^3 = U \oplus U^\perp$. To determine this we solve

$$\boldsymbol{w}_0 = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix},$$

so

$$\alpha \qquad - 3\gamma = -1 \tag{6.1}$$
$$\alpha \ + \beta + 2\gamma = 5 \tag{6.2}$$
$$\alpha - 2\beta \ + \gamma = 1. \tag{6.3}$$

Multiplying (6.2) by 2 and adding to (6.3) gives

$$3\alpha + 5\gamma = 11.$$

Then multiplying (6.1) by 3 and subtracting gives

$$14\gamma = 14.$$

Hence $\gamma = 1$, $\alpha = -1 + 3\gamma = 2$ and $\beta = 5 - \alpha - 2\gamma = 1$. We conclude

$$\boldsymbol{w}_0 = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

$$= P_U(\boldsymbol{w}_0) + \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

We know $P_U(\boldsymbol{w}_0)$ is the nearest vector in U to $\boldsymbol{w}_0$, so the distance of $\boldsymbol{w}_0$ to U is

$$\|\boldsymbol{w}_0 - P_U(\boldsymbol{w}_0)\| = \left\| \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \right\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}.$$

□

Example 6A (Exam Paper, January 2010) *Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^4$, namely*

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{4} x_i y_i$$

for $\boldsymbol{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ *and* $\boldsymbol{v} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.

(i) *Apply the Gram–Schmidt Process to the set*

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

to produce an orthonormal basis for $\mathbb{R}^4$.

(ii) *Let U be the subspace of $\mathbb{R}^4$ spanned by*

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}.$$

Find a basis for the orthogonal complement to U in $\mathbb{R}^4$.

(iii) *Find the vector in U that is nearest to* $\begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}$.

Solution: (i) Define

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We perform the steps of the Gram–Schmidt Process:

Step 1:

$$\|\boldsymbol{v}_1\|^2 = 1^2 + 1^2 + (-1)^2 + 1^2 = 4,$$

so

$$\|\boldsymbol{v}_1\| = 2.$$

Take

$$\boldsymbol{e}_1 = \frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Step 2:

$$\langle \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle = \frac{1}{2}\left\langle \begin{pmatrix} 3 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \tfrac{1}{2}(3+1+2+2) = 4.$$

Take

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$\|\boldsymbol{w}_2\|^2 = 1^2 + (-1)^2 = 2,$$

so take

$$\boldsymbol{e}_2 = \frac{1}{\|\boldsymbol{w}_2\|}\boldsymbol{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Step 3:

$$\langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle = \frac{1}{2}\left\langle \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \tfrac{1}{2}(2-4-3+1) = -2$$

$$\langle \boldsymbol{v}_3, \boldsymbol{e}_2 \rangle = \frac{1}{\sqrt{2}}\left\langle \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \tfrac{1}{\sqrt{2}}(2+4+0+0) = \frac{6}{\sqrt{2}} = 3\sqrt{2}.$$

Take

$$\begin{aligned} \boldsymbol{w}_3 &= \boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 - \langle \boldsymbol{v}_3, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2 \\ &= \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix} + 2 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}.$$

Then

$$\|\boldsymbol{w}_3\|^2 = 2^2 + 2^2 = 8,$$

so take

$$\boldsymbol{e}_3 = \frac{1}{\|\boldsymbol{w}_3\|}\boldsymbol{w}_3 = \frac{1}{2\sqrt{2}}\boldsymbol{w}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Step 4:

$$\langle \boldsymbol{v}_4, \boldsymbol{e}_1 \rangle = \frac{1}{0} \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \tfrac{1}{2}$$

$$\langle \boldsymbol{v}_4, \boldsymbol{e}_2 \rangle = \frac{1}{\sqrt{2}} \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\langle \boldsymbol{v}_4, \boldsymbol{e}_3 \rangle = \frac{1}{\sqrt{2}} \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle = 0.$$

Take

$$\begin{aligned} \boldsymbol{w}_4 &= \boldsymbol{v}_4 - \langle \boldsymbol{v}_4, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 - \langle \boldsymbol{v}_4, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2 - \langle \boldsymbol{v}_4, \boldsymbol{e}_3 \rangle \boldsymbol{e}_3 \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ -1/4 \end{pmatrix}. \end{aligned}$$

Then

$$\|\boldsymbol{w}_4\|^2 = \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2 + \left(-\tfrac{1}{4}\right)^2 = \tfrac{1}{4},$$

so take

$$\boldsymbol{e}_4 = \frac{1}{\|\boldsymbol{w}_4\|}\boldsymbol{w}_4 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is the orthonormal basis for $\mathbb{R}^4$ obtained by applying the Gram–Schmidt Process to $\mathscr{A}$.

(ii) In terms of the notation of (i), $U = \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$. However, the method of the Gram–Schmidt Process (see the proof of Theorem 6.10) shows that

$$\operatorname{Span}(\boldsymbol{e}_1, \boldsymbol{e}_2) = \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = U.$$

If $\boldsymbol{v} = \alpha\boldsymbol{e}_1 + \beta\boldsymbol{e}_2 + \gamma\boldsymbol{e}_3 + \delta\boldsymbol{e}_4$ is an arbitrary vector of $\mathbb{R}^4$ (expressed in terms of our orthonormal basis), then

$$\langle \boldsymbol{v}, \boldsymbol{e}_1 \rangle = \alpha \qquad \text{and} \qquad \langle \boldsymbol{v}, \boldsymbol{e}_2 \rangle = \beta.$$

Hence if $\boldsymbol{v} \in U^{\perp}$, then in particular $\alpha = \beta = 0$, so $U^{\perp} \subseteq \operatorname{Span}(\boldsymbol{e}_3, \boldsymbol{e}_4)$. Conversely, if $\boldsymbol{v} = \gamma\boldsymbol{e}_3 + \delta\boldsymbol{e}_4 \in \operatorname{Span}(\boldsymbol{e}_3, \boldsymbol{e}_4)$, then

$$\langle \zeta\boldsymbol{e}_1 + \eta\boldsymbol{e}_2, \gamma\boldsymbol{e}_3 + \delta\boldsymbol{e}_4 \rangle = 0$$

since $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = 0$ for $i \neq j$. Hence every vector in $\operatorname{Span}(\boldsymbol{e}_3, \boldsymbol{e}_4)$ is orthogonal to every vector in U and we conclude

$$U^{\perp} = \operatorname{Span}(\boldsymbol{e}_3, \boldsymbol{e}_4).$$

Thus $\{\boldsymbol{e}_3, \boldsymbol{e}_4\}$ is a basis for $U^{\perp}$.

(iii) Let $P \colon V \to V$ be the projection onto U associated to the direct sum decomposition $V = U \oplus U^{\perp}$. Then $P(\boldsymbol{v})$ is the vector in U closest to v. Now in our application of the Gram–Schmidt Process,

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 - \langle \boldsymbol{v}_3, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2,$$

so

$$P(\boldsymbol{w}_3) = P(\boldsymbol{v}_3) - \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle P(\boldsymbol{e}_1) - \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle P(\boldsymbol{e}_2).$$

Therefore

$$\boldsymbol{0} = P(\boldsymbol{v}_3) - \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 - \langle \boldsymbol{v}_3, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2,$$

since $\boldsymbol{w}_3 = \|\boldsymbol{w}_3\|\boldsymbol{e}_3 \in U^{\perp}$ and $\boldsymbol{e}_1, \boldsymbol{e}_2 \in U$. Hence the closest vector in U to $\boldsymbol{v}_3$ is

$$\begin{aligned} P(\boldsymbol{v}_3) &= \langle \boldsymbol{v}_3, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 + \langle \boldsymbol{v}_3, \boldsymbol{e}_2 \rangle \boldsymbol{e}_2 \\ &= (-2) \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= -\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -4 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

□

We finish this section by demonstrating an application of this projection map that is used in many other areas of mathematics (for example, in the analysis of data in statistics). We start with the following observation:

Proposition 6.19 *Let A be a real $m \times n$ matrix and suppose that $\ker A = \{\mathbf{0}\}$. Then*

(i) *the matrix $A^{\mathsf{T}}A$ is invertible;*

(ii) *the matrix $P = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}$ is the projection map onto the image $U = \operatorname{im} A$ associated to the direct sum decomposition $\mathbb{R}^m = U \oplus U^{\perp}$;*

(iii) *if $\boldsymbol{v} \in \mathbb{R}^m$, then $A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\boldsymbol{v}$ is the closest vector in $\operatorname{im} A$ to $\boldsymbol{v}$.*

Recall that the *transpose* of a matrix is obtained by interchanging rows and columns. Thus if $A = [a_{ij}]$ is an $m \times n$ matrix, then $A^{\mathsf{T}} = [a_{ji}]$ is the $n \times m$ matrix whose (i,j)th entry is a_{ji} (the (j,i)th entry of A). Since the inner product on $\mathbb{R}^n$ is the dot product, the transpose is relevant since, for

$$\boldsymbol{u} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

the inner product is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} x_i y_i = \boldsymbol{u}^{\mathsf{T}}\boldsymbol{v}$$

(the latter being the product of the $1 \times n$ matrix $\boldsymbol{u}^{\mathsf{T}}$ with the $n \times 1$ matrix $\boldsymbol{v}$). We shall make use of the standard formulae

$$(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$$

for any suitable matrices A and B. (This is established by straightforward calculations, but also see Question 2 on Problem Sheet VI.)

The hypothesis that $\ker A = \{\mathbf{0}\}$ (that is, A has nullity 0) ensures that the columns of A are linearly independent vectors and so $\operatorname{rank} A = n$. (See Proposition 2.9 for the relevant background.) Let us now establish this proposition before discussing its application.

PROOF: (i) Since A is an $m \times n$ matrix, the transpose A^{T} is an $n \times m$ matrix and so the product $A^{\mathsf{T}}A$ is an $n \times n$ matrix. Suppose $\boldsymbol{v} \in \ker A^{\mathsf{T}}A$, so $A^{\mathsf{T}}A\boldsymbol{v} = \mathbf{0}$. Then

$$\langle A\boldsymbol{v}, A\boldsymbol{v} \rangle = (A\boldsymbol{v})^{\mathsf{T}}A\boldsymbol{v} = \boldsymbol{v}^{\mathsf{T}}A^{\mathsf{T}}A\boldsymbol{v} = \boldsymbol{v}^{\mathsf{T}}\mathbf{0} = \langle \boldsymbol{v}, \mathbf{0} \rangle = 0.$$

Hence $A\boldsymbol{v} = \mathbf{0}$, so $\boldsymbol{v} = \mathbf{0}$ since $\ker A = \{\mathbf{0}\}$. This shows that $\ker A^{\mathsf{T}}A = \{\mathbf{0}\}$ and therefore $A^{\mathsf{T}}A$ is invertible (as it is a square matrix).

(ii) Since $A^{\mathsf{T}}A$ is an invertible $n \times n$ matrix, we may form $P = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}$ and this is an $m \times m$ matrix. Let $\boldsymbol{v} \in \mathbb{R}^m$. Then

$$P\boldsymbol{v} = A\big((A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\boldsymbol{v}\big) \in \operatorname{im} A.$$

Consider $\boldsymbol{y} = \boldsymbol{v} - P\boldsymbol{v} \in \mathbb{R}^m$. If $\boldsymbol{w}$ is any vector in $\mathbb{R}^n$, then

$$\langle A\boldsymbol{w}, \boldsymbol{y} \rangle = \langle A\boldsymbol{w}, \boldsymbol{v} \rangle - \langle A\boldsymbol{w}, P\boldsymbol{v} \rangle$$

$$
\begin{aligned}
&= \boldsymbol{w}^{\mathsf{T}} A^{\mathsf{T}} \boldsymbol{v} - \boldsymbol{w}^{\mathsf{T}} A^{\mathsf{T}} A (A^{\mathsf{T}} A)^{-1} A^{\mathsf{T}} \boldsymbol{v} \\
&= \boldsymbol{w}^{\mathsf{T}} A^{\mathsf{T}} \boldsymbol{v} - \boldsymbol{w}^{\mathsf{T}} A^{\mathsf{T}} \boldsymbol{v} = 0.
\end{aligned}
$$

Hence $\boldsymbol{y}$ is orthogonal to $A\boldsymbol{w}$ for every $\boldsymbol{w} \in \mathbb{R}^n$; that is, $\boldsymbol{y} \in (\operatorname{im} A)^{\perp}$. Now

$$
\boldsymbol{v} = P\boldsymbol{v} + (v - P\boldsymbol{v}) = P\boldsymbol{v} + \boldsymbol{y}
$$

expresses $\boldsymbol{v}$ as the sum of a vector in $\operatorname{im} A$ and a vector in its orthogonal complement. This must therefore be the unique expression for $\boldsymbol{v}$ in terms of the direct sum decomposition $\mathbb{R}^m = \operatorname{im} A \oplus (\operatorname{im} A)^{\perp}$. Consequently $P\boldsymbol{v}$ is the image of $\boldsymbol{v}$ under the projection map onto $\operatorname{im} A$ associated to this direct sum decomposition and now, by Theorem 6.17, $P\boldsymbol{v}$ is the closest vector in $\operatorname{im} A$ to $\boldsymbol{v}$. $\square$

Example 6.20 (Linear Least Squares Approximation) Suppose that we are trying to determine how some real-valued function $y = y(t)$ depends on the real variable t. In this example, we shall fit a linear model, so approximate y by a linear expression

$$
y(t) \approx \alpha t + \beta
$$

for some constants α and β. To determine the best choice of α and β, we make a (usually large) number of measurements of the value of y for a variety of inputs t and so find a sequence of pairs

$$
(t_1, y_1),\ (t_2, y_2),\ \ldots,\ (t_m, y_m)
$$

that comprises our data. We seek to find scalars α and β such that the line $\hat{y}(t) = \alpha t + \beta$ is the best fit to pass through the scattered points of data; that is, such that $A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is as close as possible to $\boldsymbol{y}$, where

$$
A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_m & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.
$$

If it were the case that y was precisely given by a formula $y = \alpha t + \beta$ and we were able to measure y exactly, then $\boldsymbol{y}$ would be in the image of A and we would simply find the pair $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ that maps to $\boldsymbol{y}$. In such an idealized situation, the points in the data would actually lie on a line and we are merely seeking to find its parametrization. In real life, however, we intend to approximate y by a linear function (so that we can exploit the methods of linear mathematics) and any measurement of y will come with some error. In this case, $\boldsymbol{y}$ will most likely not be in the image of A and so we seek to make $A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ as close to $\boldsymbol{y}$ as possible. To justify the name of the application, note that

$$
\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}
$$

and so finding the vector $\boldsymbol{x}$ in $\operatorname{im} A$ closest to $\boldsymbol{y}$ does correspond to minimizing the expression

$$
\sum_{i=1}^{m} (x_i - y_i)^2
$$

(i.e., minimizing this sum of squares).

Provided the values t_i are distinct, then the two columns of A are linearly independent; that is, $\operatorname{rank} A = 2$ and $\operatorname{null} A = 0$. We may then apply Proposition 6.19 and Part (iii) tells us what this closest vector $\boldsymbol{x}$ in $\operatorname{im} A$ to $\boldsymbol{y}$ is:

$$\boldsymbol{x} = P\boldsymbol{y} = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\boldsymbol{y}$$

and $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is the unique vector satisying $A\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \boldsymbol{x}$; that is,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\boldsymbol{y}.$$

In practice, one would like to take a very large number of measurements and then computer calculations are used to determine the scalars α and β.

Chapter 7

The adjoint of a transformation and self-adjoint transformations

In Proposition 6.19, we observed that the transpose of a matrix was significant. In this chapter, we shall explain this further by introducing the adjoint of a linear transformation.

Throughout this section, V is a finite-dimensional inner product space over a field F (where, as before, $F = \mathbb{R}$ or $\mathbb{C}$) with inner product $\langle \cdot, \cdot \rangle$.

Definition 7.1 Let $T \colon V \to V$ be a linear transformation. The *adjoint* of T is a map $T^* \colon V \to V$ such that

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle \qquad \text{for all } v, w \in V.$$

Remark: More generally, if $T \colon V \to W$ is a linear map between inner product spaces, the adjoint $T^* \colon W \to V$ is a map satisfying the above equation for all $v \in V$ and $w \in W$. Appropriate parts of what we describe here can be done in this more general setting.

Lemma 7.2 *Let V be a finite-dimensional inner product space and let $T \colon V \to V$ be a linear transformation. Then there is a unique adjoint T^* of T and, moreover, T^* is a linear transformation.*

PROOF: We first show that if T^* exists, then it is unique. Suppose that $S \colon V \to V$ also satisfies the same condition. Then

$$\langle v, T^*(w) \rangle = \langle T(v), w \rangle = \langle v, S(w) \rangle \qquad \text{for all } v, w \in V.$$

Hence

$$\langle v, T^*(w) \rangle - \langle v, S(w) \rangle = 0;$$

that is,

$$\langle v, T^*(w) - S(w) \rangle = 0 \qquad \text{for all } v, w \in V.$$

Let us fix $w \in V$ and take $v = T^*(w) - S(w)$. Then

$$\langle T^*(w) - S(w), T^*(w) - S(w) \rangle = 0.$$

The axioms of an inner product space tell us

$$T^*(w) - S(w) = \mathbf{0}$$

so

$$S(w) = T^*(w) \qquad \text{for all } w \in V.$$

This shows that if the adjoint T^* exists, then it is unique.

It remains to show that such a linear map T^* actually exists. Let $\mathscr{B} = \{e_1, e_2, \dots, e_n\}$ be an orthonormal basis for V. (The Gram–Schmidt Process guarantees that this exists.) Let $A = [\alpha_{ij}]$ be the matrix of T with respect to $\mathscr{B}$. Define $T^*\colon V \to V$ be the linear map whose matrix is the conjugate transpose of A with respect to $\mathscr{B}$. Thus

$$T^*(e_j) = \sum_{i=1}^{n} \bar{\alpha}_{ji} e_i \qquad \text{for } j = 1, 2, \dots, n.$$

(Here we are using Proposition 2.7 to guarantee that this determines a unique linear transformation T^*.) Note also that

$$T(e_j) = \sum_{i=1}^{n} \alpha_{ij} e_i \qquad \text{for } j = 1, 2, \dots, n.$$

Claim: $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$ for all $v, w \in V$.

Write $v = \sum_{j=1}^{n} \beta_j e_j$ and $w = \sum_{k=1}^{n} \gamma_k e_k$ in terms of the basis $\mathscr{B}$. Then

$$\begin{aligned}
\langle T(v), w\rangle &= \left\langle T\left(\sum_{j=1}^{n} \beta_j e_j\right), \sum_{k=1}^{n} \gamma_k e_k \right\rangle \\
&= \left\langle \sum_{j=1}^{n} \beta_j T(e_j), \sum_{k=1}^{n} \gamma_k e_k \right\rangle \\
&= \left\langle \sum_{j=1}^{n} \beta_j \sum_{i=1}^{n} \alpha_{ij} e_i, \sum_{k=1}^{n} \gamma_k e_k \right\rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} \beta_j \alpha_{ij} \bar{\gamma}_k \langle e_i, e_k\rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \beta_j \alpha_{ij} \bar{\gamma}_i,
\end{aligned}$$

while

$$\begin{aligned}
\langle v, T^*(w)\rangle &= \left\langle \sum_{j=1}^{n} \beta_j e_j, T^*\left(\sum_{k=1}^{n} \gamma_k e_k\right) \right\rangle \\
&= \left\langle \sum_{j=1}^{n} \beta_j e_j, \sum_{k=1}^{n} \gamma_k T^*(e_k) \right\rangle \\
&= \left\langle \sum_{j=1}^{n} \beta_j e_j, \sum_{k=1}^{n} \gamma_k \sum_{i=1}^{n} \bar{a}_{ki} e_i \right\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{i=1}^{n} \beta_j \bar{\gamma}_k a_{ki} \langle e_j, e_i\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_j \bar{\gamma}_k a_{kj}
\end{aligned}$$

$$= \langle T(v), w \rangle.$$

Hence T^* is indeed the adjoint of T. □

We also record what was observed in the course of this proof:

Lemma 7.3 *If $A = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ is the matrix of the linear map $T \colon V \to V$ with respect to an orthonormal basis $\mathscr{B}$ for V, then*

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T^*) = \bar{A}^{\mathsf{T}}$$

(the conjugate transpose *of A).* □

Diagonalisation of self-adjoint transformations

Definition 7.4 A linear transformation $T \colon V \to V$ is *self-adjoint* if $T^* = T$.

Interpreting this in terms of the matrices (using Lemma 7.3 above), we conclude:

Lemma 7.5 (i) *A real matrix A defines a self-adjoint transformation if and only if it is* symmetric*: $A^{\mathsf{T}} = A$.*

(ii) *A complex matrix A defines a self-adjoint transformation if and only if it is* Hermitian*: $\bar{A}^{\mathsf{T}} = A$.* □

We shall establish the following important theorem:

Theorem 7.6 (Spectral Theorem) *A self-adjoint transformation of a finite-dimensional inner product space is diagonalisable.*

Interpreting this in terms of matrices gives us:

Corollary 7.7 (i) *A real symmetric matrix is diagonalisable.*

(ii) *A Hermitian matrix is diagonalisable.*

To establish Theorem 7.6, we shall start by establishing the main tools needed to prove that result.

Lemma 7.8 *Let V be a finite-dimensional inner product space and $T \colon V \to V$ be a self-adjoint transformation. Then the characteristic polynomial is a product of linear factors and every eigenvalue of T is real.*

Proof: Any polynomial is factorisable over $\mathbb{C}$ into a product of linear factors. Thus it is sufficient to show all the roots of the characteristic polynomial are real.

Let W be an inner product space over $\mathbb{C}$ with the same dimension as V and let $S \colon W \to W$ be a linear transformation whose matrix A with respect to an orthonormal basis for W is the same as that of T with respect to an orthonormal basis for V. Since $T^* = T$, the corresponding matrices are equal: $\bar{A}^{\mathsf{T}} = A$. Hence S is also self-adjoint. (Essentially this process is used to deal with the fact that V might be a vector space over $\mathbb{R}$, so we replace it by an inner product space over $\mathbb{C}$ that in all other ways is the same.)

Let $\lambda \in \mathbb{C}$ be a root of $c_S(x) = \det(xI - A) = c_T(x)$. Then λ is an eigenvalue of S, so there exists an eigenvector $v \in W$ for S:

$$S(v) = \lambda v.$$

Therefore

$$\langle S(v), v\rangle = \langle \lambda v, v\rangle = \lambda\|v\|^2,$$

but also

$$\langle S(v), v\rangle = \langle v, S^*(v)\rangle = \langle v, S(v)\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\|v\|^2.$$

Hence

$$\lambda\|v\|^2 = \bar{\lambda}\|v\|^2$$

and since $v \neq \mathbf{0}$, we conclude $\lambda = \bar{\lambda}$. This shows that $\lambda \in \mathbb{R}$ and the lemma is proved. □

Lemma 7.9 *Let V be an inner product space and $T\colon V \to V$ be a linear map. If U is a subspace of V such that $T(U) \subseteq U$ (that is, U is T-*invariant*), then $T^*(U^\perp) \subseteq U^\perp$ (that is, $U^\perp$ is T^*-invariant).*

Proof: Let $v \in U^\perp$. Then, for any $u \in U$,

$$\langle u, T^*(v)\rangle = \langle T(u), v\rangle = 0,$$

since $T(u) \in U$ (by assumption) and $v \in U^\perp$. Hence $T^*(v) \in U^\perp$. □

These two lemmas now enable us to prove the main theorem about diagonalisation of self-adjoint transformations.

Proof of Theorem 7.6: Let $T\colon V \to V$ be a self-adjoint linear map. We proceed by induction on $n = \dim V$. If $n = 1$, then T is represented by a 1×1 matrix, which is already diagonal.

Assume then that $n > 1$ and that the result holds for self-adjoint linear transformations of inner product spaces of dimension smaller than n. Consider the characteristic polynomial $c_T(x)$. By Lemma 7.8, this is a product of linear factors. In particular, there exists some root $\lambda \in F$. Let v_1 be an eigenvector with eigenvalue λ. Let $U = \mathrm{Span}(v_1)$ be the 1-dimensional subspace spanned by v_1. By Theorem 6.16,

$$V = U \oplus U^\perp.$$

Now as $T(v_1) = \lambda v_1 \in U$, we see that U is T-invariant. Hence $U^\perp$ is also T-invariant by Lemma 7.9 (since $T^* = T$).

Now consider the restriction $S = T|_{U^\perp}\colon U^\perp \to U^\perp$ of T to $U^\perp$. This is self-adjoint, since

$$\langle T(v), w\rangle = \langle v, T(w)\rangle \qquad \text{for all } v, w, \in U^\perp$$

tells us

$$(T|_{U^\perp})^* = T|_{U^\perp}.$$

By induction, $S = T|_{U^\perp}$ is diagonalisable. Hence there is a basis $\{v_2, \dots, v_n\}$ for $U^\perp$ of eigenvectors for T. Then as $V = U \oplus U^\perp$, we conclude that $\{v_1, v_2, \dots, v_n\}$ is a basis for V consisting of eigenvectors for T. Hence T is diagonalisable and the proof is complete. □

We finish the course by making a number of observations about what diagonalisation of a self-adjoint linear map achieves for us. We start by making the following observation.

Lemma 7.10 *Let V be an inner product space and $T\colon V \to V$ be a self-adjoint linear map. Then any pair of eigenvectors for distinct eigenvalues are orthogonal.*

The observation in this lemma is actually implicit in the proof of Theorem 7.6 at the point where we passed from the vector space V to the orthogonal complement of the subspace spanned by an eigenvector.

Proof: Let u and v be eigenvectors for T with eigenvalues λ and μ, respectively, with $\lambda \neq \mu$. By Lemma 7.8, λ and μ are real numbers. Since $T^* = T$ and $\bar{\mu} = \mu$,

$$\lambda\langle u, v\rangle = \langle \lambda u, v\rangle = \langle T(u), v\rangle = \langle u, T^*(v)\rangle = \langle u, T(v)\rangle = \langle u, \mu v\rangle = \mu\langle u, v\rangle$$

and so

$$(\lambda - \mu)\langle u, v\rangle = 0.$$

As $\lambda - \mu \neq 0$, we conclude $\langle u, v\rangle = 0$; that is, u and v are orthogonal. □

Let us now interpret what we have achieved in Theorem 7.6. Let $T \colon V \to V$ be a self-adjoint linear transformation of an inner product space V. By the theorem, T is diagonalisable and therefore there is a basis for V consisting of eigenvectors for T. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the eigenvalues of T. The vector space V is then the direct sum of the eigenspaces

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k} \tag{7.1}$$

and, by Lemma 7.10, any vector in E_{λ_i} is orthogonal to any vector in E_{λ_j} when $i \neq j$. Therefore

$$E_{\lambda_i}^{\perp} = \bigoplus_{j \neq i} E_{\lambda_j}$$

and the above direct sum (7.1) is a decomposition of V into orthogonal spaces. For each $i = 1, 2, \dots, k$, let $P_i \colon V \to V$ be the projection map onto the ith summand E_{λ_i}. If $v = v_1 + v_2 + \dots + v_k$ with $v_i \in E_{\lambda_i}$ for each i, then $P_i(v) = v_i$ by definition and $T(v_i) = \lambda_i v_i$ since v_i lies in the eigenspace E_{λ_i}. Hence

$$\begin{aligned} T(v) &= T(v_1) + T(v_2) + \dots + T(v_k) \\ &= \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k \\ &= \lambda_1 P_1(v) + \lambda_2 P_2(v) + \dots + \lambda_k P_k(v) \end{aligned}$$

for all $v \in V$ and so

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_k P_k.$$

Furthermore, for v as just expressed,

$$v = v_1 + v_2 + \dots + v_k = P_1(v) + P_2(v) + \dots + P_k(v)$$

and so

$$P_1 + P_2 + \dots + P_k = I.$$

Let $u = u_1 + u_2 + \dots u_k$ and $v = v_1 + v_2 + \dots v_k$ be the decompositions of two vectors u and v. Since vectors in E_{λ_i} are orthogonal to vectors in E_{λ_j} when $i \neq j$,

$$\langle P_i(u), v\rangle = \langle u_i, v_1 + v_2 + \dots + v_k\rangle = \langle u_i, v_i\rangle$$

and

$$\langle u, P_i(v)\rangle = \langle u_1 + u_2 + \dots + u_k, v_i\rangle = \langle u_i, v_i\rangle$$

for any i. Therefore

$$\langle P_i(u), v\rangle = \langle u, P_i(v)\rangle \qquad \text{for all } u, v \in V$$

so $P_i^* = P_i$.

The above observations, together with the properties of projection maps established in Chapter 3, give the final result of the course:

Theorem 7.11 (Spectral Decomposition) *Let T be a self-adjoint transformation of an inner product space V. Let λ_1, λ_2, …, λ_k be the distinct eigenvalues of T. Then*

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

is the orthogonal direct sum of the eigenspaces and, if P_i denotes the associated projection map onto E_{λ_i}, then

(i) $P_1 + P_2 + \cdots + P_k = I$;

(ii) $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k = T$;

(iii) $P_i^2 = P_i = P_i^*$ *for* $i = 1, 2, \ldots, k$;

(iv) $P_i P_j = 0$ *if* $i \neq j$. □

This spectral decomposition applies, in particular, to a real symmetric matrix and to a complex Hermitian matrix.

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Versions

Significant updates to the notes will be listed below. Updates that are merely correcting typographic errors and similar will be indicated by appending to a letter to the version number on the front page and will not be listed below.

Version 0.9: First released version. All material to be covered is present in the notes (but some more detailed cross-referencing to problem sheets yet to be implemented).

Version 1.0: Simplified proof of Proposition 6.19. Updated references in the notes to questions on problem sheets. Fixed three typos in Chapter 1.

Version 1.1: Minor corrections to the statement of Proposition 6.19.