School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet IX: Cosets; Lagrange's Theorem

1. Consider the dihedral group D_{12} of order 12 consisting of isometries of the regular hexagon.
   (a) Let σ be one of the reflections of the hexagon. What is the index of the subgroup $H = \langle \sigma \rangle$ in D_{12}? Determine whether or not every right coset of H in D_{12} is also a left coset.
   (b) Find a subgroup of order 3 in D_{12} and list its right cosets.
   (c) How many subgroups of order 5 are there in D_{12}?

2. Let U_{15} denote the group of congruence classes that have multiplicative inverses modulo 15:
$$U_{15} = \{\, x \in \mathbb{Z}/15\mathbb{Z} \mid \gcd(x, 15) = 1 \,\} = \{1, 2, 4, 7, 8, 11, 13, 14\}$$
   (a) Show that each of the following subsets is a subgroup of U_{15}:
$$H_1 = \{1, 2, 4, 8\}; \qquad H_2 = \{1, 4, 7, 13\};$$
$$H_3 = \{1, 4\}; \qquad H_4 = \{1, 11\}.$$
   (b) Find the left and right cosets of each of the subgroups H_1, H_2, H_3 and H_4 in U_{15}.

3. Let
$$H = \{(\,), (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$$
$$K = \{(\,), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$
   You may assume that H and K are subgroups of the symmetric group S_4 of degree 4.
   Find the left and right cosets of both H and K in S_4.
   Do the left and right cosets of each of these subgroups coincide?

4. Suppose that G is a finite group with subgroups H and K satisfying $K \leqslant H \leqslant G$. Show that
$$|G : K| = |G : H| \cdot |H : K|.$$

5. (a) Let G be a finite group. Show that $x^{|G|} = 1$ for all $x \in G$.
   (b) Prove **Fermat's Little Theorem**: If p is a prime, show that $x^{p-1} \equiv 1 \pmod{p}$ for all integers x that are not divisible by p.
   (c) Calculate
$$2017^{13} \pmod{13} \qquad \text{and} \qquad 13^{2017} \pmod{2017}.$$

6. Let $G = U_{16} = \{1, 3, 5, 7, 9, 11, 13, 15\}$, the group of congruence classes with multiplicative inverses modulo 16. Let

$$H = \{1, 7, 9, 15\} \qquad \text{and} \qquad K = \{1, 7\}.$$

Show that H and K are subgroups of G.

Find the cosets of H in G, find the cosets of K in H, and the cosets of K in G.

7. Let G be a group and H be a subgroup of G.

Define a relation $\sim$ on G by

$$x \sim y \qquad \text{when } xy^{-1} \in H.$$

(a) Show that $\sim$ is an equivalence relation on G.

(b) If $x \in G$, show that the equivalence class of x under the relation $\sim$ equals the right coset Hx.

8. Let G be a group and H be a subgroup of G.

(a) If $x, y \in G$, show that $xH = yH$ if and only if $x^{-1}y \in H$.

(b) Show that two left cosets of H in G are either equal or disjoint.

(c) Show that G is the disjoint union of the left cosets of H.

(d) Show that every left coset of H in G contains the same number of elements as the subgroup of H (that is, show $|xH| = |H|$ for all $x \in G$).

9. Let G be a group and H be a subgroup of G. Show that there is a bijection f from the set of right cosets of H in G to the set of left cosets of H in G given by

$$f\colon Hx \mapsto x^{-1}H \qquad \text{for } x \in G.$$

[Note: You should, in particular, demonstrate that the function f is *well-defined*; that is, the value $f(Hx)$ does not depend upon the choice of representative x. To do this one must show that if $Hx = Hy$ then necessarily $f(Hx) = f(Hy)$.]

10. (a) Suppose that G is a finite group of odd order. Show that every element of G has odd order.

Show that every element x of G is a square; that is, there exists $y \in G$ with $y^2 = x$.

(b) Suppose that G is a finite group of even order. Show that there exists a non-identity element of order 2.

Show that there is an element of G which is not a square.

11. Show that the alternating group A_4 of degree 4 contains no subgroup of order 6.

[Hint: First list the 12 elements of A_4. If H is a subgroup containing six elements, what forms of element must it necessarily contain? Calculate some products and obtain a contradiction. (As an aside, there is a shorter argument if one understands quotient groups — from the final chapter of the module — but the question can be solved by the proposed more elementary method.)]