School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet VIII: Subgroups; cyclic subgroups; the order of an element; alternating groups

1. Determine which of the following subsets are subgroups of the given group:
   (a) the subset $\{0, 1, 4, 6\}$ of $\mathbb{Z}/7\mathbb{Z}$ under addition modulo 7;
   (b) the subset $\{1, 2, 4\}$ of the multiplicative group $\mathbb{F}_7^*$ of non-zero congruence classes modulo 7;
   (c) the subset $\{1, 4, 7\}$ of the group $U_9 = \{1, 2, 4, 5, 7, 8\}$ under multiplication modulo 9;
   (d) the subset $\{(\,),\ (1\ 2),\ (2\ 3),\ (1\ 2\ 3)\}$ of the symmetric group S_3 of degree 3;
   (e) the subset of all rotations of the dihedral group D_8 of order 8.

2. Let $G = \{\,(a, b) \mid a, b \in \mathbb{R}, a \neq 0\,\}$ and define
$$(a, b)(c, d) = (ac, bc + d).$$
   Show that G is a non-abelian group under this operation.
   Determine which of the following subsets are subgroups of G:
   (a) $H_k = \{\,(a, k(a - 1)) \mid a \neq 0\,\}$, where k is a fixed real number;
   (b) $K = \{\,(a, 0) \mid a > 0\,\}$;
   (c) $L_m = \{\,(a, ma^m) \mid a \neq 0\,\}$, where m is a fixed integer;
   (d) $M = \{\,(1, b) \mid b \in \mathbb{R}\,\}$.

3. Let $G = \{\,(a, b) \mid a, b \in \mathbb{R}\,\}$ and define
$$(a, b)(c, d) = (a + c, b + 2^a d).$$
   Show that G is a non-abelian group under this operation.
   Show that each of the following subsets is a subgroup of G:
   (a) $H = \{\,(a, 0) \mid a \in \mathbb{R}\,\}$;
   (b) $K = \{\,(0, b) \mid b \in \mathbb{R}\,\}$.

4. Let X be a figure in the plane and consider the set
$$\operatorname{Isom}(X) = \{\, f \in \operatorname{Isom}(\mathbb{R}^2) \mid Xf = X\,\}$$
   of isometries of $\mathbb{R}^2$ that map the figure X to itself. Show that $\operatorname{Isom}(X)$ is a subgroup of the group $\operatorname{Isom}(\mathbb{R}^2)$ of all isometries of $\mathbb{R}^2$.
   [This verifies that $\operatorname{Isom}(X)$ is itself a group.]

5. Let G be a group and let H and K be subgroups of G.

   (a) Show that the intersection $H \cap K$ is also a subgroup of G.

   (b) Show that the union $H \cup K$ is a subgroup of G if and only if $H \leqslant K$ or $K \leqslant H$.

   (c) Suppose that $\{H_1, H_2, \dots\}$ is an infinite collection of subgroups of G such that

   $$H_1 \leqslant H_2 \leqslant \dots.$$

   Show that $\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G.

   (d) Is there an example of a group G with three subgroups H, K and L, none of which are contained in any of the others, such that $H \cup K \cup L$ is a subgroup of G?

6. Consider the additive group $\mathbb{Z}$ of integers. Suppose that A and B are non-trivial subgroups of $\mathbb{Z}$; that is, $A, B \neq \{0\}$. Show that $A \cap B$ is infinite.

7. (a) Determine the elements in the cyclic subgroup of the symmetric group S_8 generated by the permutation $(1\;2\;3\;4)\,(5\;6\;7\;8)$.

   (b) Determine the elements in the cyclic subgroup generated by 11 in the additive group $\mathbb{Z}/24\mathbb{Z}$ (under addition modulo 24).

8. For each of the following groups, determine whether or not it is cyclic:

   (a) $U_8 = \{1, 3, 5, 7\}$ under multiplication modulo 8;

   (b) $U_9 = \{1, 2, 4, 5, 7, 8\}$ under multiplication modulo 9;

   (c) $\{1, 3, 9, 11\}$ under multiplication modulo 16;

   (d) $\{1, 7, 9, 15\}$ under multiplication modulo 16;

   (e) the symmetric group S_3 of degree 3;

   (f) the Klein 4-group.

9. Let $G = \langle x \rangle$ be a cyclic group. Show that any subgroup H of G is also cyclic.

   [Hint: If $H \neq \{1\}$, show that H contains x^m for some $m > 0$ and then consider the subgroup generated by x^k where k is chosen to the smallest positive integer with $x^k \in H$.]

10. For each of the following groups, list the elements of the group and determine the order of each element:

    (a) the symmetric group S_4 of degree 4;

    (b) the dihedral group D_{12} of order 12 (consisting of isometries of a regular hexagon);

    (c) the multiplicative group $\mathbb{F}_7^*$ of non-zero congruence classes modulo 7.

11. Let G be a group and let $a, b \in G$ be elements such that $ab = ba$. Suppose that a has order m and b has order n (for some positive integers m and n). If $\gcd(m, n) = 1$, show that ab has order mn.

12. Define the following eight 2×2 matrices with entries from the complex numbers $\mathbb{C}$:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad -E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad -I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad -J = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad -K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

(a) Show that the set Q of these eight matrices is a non-abelian group under matrix multiplication.

(b) Determine the order of each matrix in Q.

13. For each of the following permutations, determine whether it is even or odd.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$

(b) $(1\,2\,4)\,(3\,6)$

(c) $(1\,6\,4)\,(1\,3\,4\,2\,5)$

(d) $(1\,2\,4\,7)\,(3\,8\,6\,5)$

14. Let σ and τ be permutations in the symmetric group S_n of degree n.

(a) Show that σ is even if and only if σ^{-1} is even.

(b) Show that $\tau^{-1}\sigma\tau$ is even if and only if σ is even.

15. We shall say that two permutations (in some symmetric group S_n) have the same *cycle structure* if, when written as products of disjoint cycles, they have the same number of cycles of each length.

Determine the possible cycle structures of permutations in the symmetric group S_5 of degree 5.

For each cycle structure, determine (i) the number of permutations with this cycle structure, (ii) the order of the permutation, and (iii) whether it is even or odd.

16. Find a subgroup of order 4 in the alternating group A_4 of degree 4.

What are the possible orders of elements of A_4?