MRQ *2024*

# School of Mathematics and Statistics

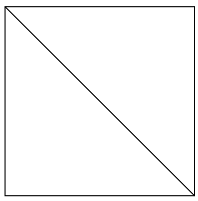
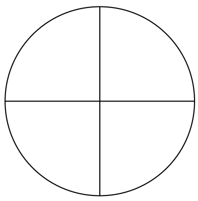
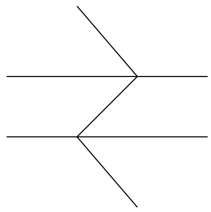
## MT2505 Abstract Algebra

## Problem Sheet VII: Isometries and Dihedral Groups

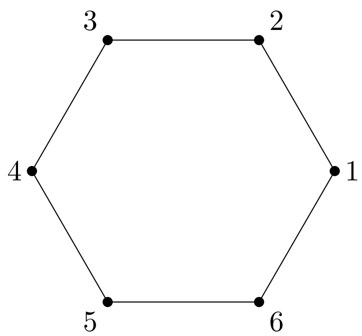
1. (a) Show that if $f$ and $g$ are isometries of $\mathbb{R}^2$, then the composite $fg$ and the inverse $f^{-1}$ are also isometries.

   (b) Show that the set $\mathrm{Isom}(\mathbb{R}^2)$ of isometries of $\mathbb{R}^2$ is a group under the operation of composition.

2. For each of the following figures $X$, describe all isometries of $X$ and construct the Cayley table of the isometry group $\mathrm{Isom}(X)$:

3. The dihedral group $D_{12}$ of order 12 consists of all isometries of the regular hexagon. Label the vertices of the hexagon with the integers 1, 2, 3, 4, 5 and 6:




   Each isometry $f$ of the hexagon determines a permutation of $\{1, 2, 3, 4, 5, 6\}$: this permutation maps $i$ to $j$ if the isometry $f$ moves the vertex labelled $i$ to the vertex labelled $j$.

   List all twelve isometries of the regular hexagon. For each isometry, write down the corresponding permutation of $\{1, 2, 3, 4, 5, 6\}$ and express it as a product of disjoint cycles.

4. Let $n$ be an integer with $n \geqslant 3$ and consider the dihedral group $D_{2n}$ of order $2n$ consisting of all isometries of the regular polygon with $n$ sides. Let $\rho$ denote a rotation about the centre through an angle of $2\pi/n$ and let $\sigma$ denote a fixed reflection about a valid axis.

   Fix integers $r, s \in \mathbb{Z}$ and let $\alpha = \rho^r\sigma$ and $\beta = \rho^s\sigma$. (These are two of the reflections of the polygon.)

   (a) Suppose that $g = g_1g_2 \dots g_k$ is a product such that $g_1, g_2, \dots, g_k \in \{\alpha, \beta\}$.
   Show that $g$ is a rotation if and only if $k$ is even.
   Furthermore, if $k$ is even, show that $g$ is a power of $\rho^{r-s}$.
   [Hint: It may help to calculate the product $\alpha\beta$ and to use the formula $\sigma\rho = \rho^{-1}\sigma$.]

   (b) Show that every element of $D_{2n}$ can be written as a product involving $\alpha$ and $\beta$ only if and only if $\gcd(r - s, n) = 1$.