School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet VI: Permutations and Symmetric Groups

1. For each of the following functions, determine whether it is injective, surjective, and/or bijective.

   (a) $f\colon \mathbb{R} \setminus \{\, n\pi/2 \mid n \in \mathbb{Z} \,\} \to \mathbb{R}$ given by $xf = \tan x$;

   (b) $f\colon \mathbb{R} \to [-1, 1]$ given by $xf = \sin x$;

   (c) $f\colon \mathbb{R} \to \mathbb{R}$ given by $xf = x^3$;

   (d) $f\colon \mathbb{R} \to \mathbb{R}$ given by $xf = 3x + 5$;

   (e) $f\colon [0, \infty) \to \mathbb{R}$ given by $xf = \sqrt{x}$.

2. For the following sets and proposed binary operation, determine whether or not this defines a group. For each, provide either a proof or an explanation of why this does not define a group.

   (a) The four functions $\{e, f, g, h\}$ from $\mathbb{R} \setminus \{0\}$ to itself given by

   $$e\colon x \mapsto x \qquad f\colon x \mapsto -x$$
   $$g\colon x \mapsto 1/x \qquad h\colon x \mapsto -1/x$$

   under composition of functions.

   (b) The set of all functions $X \to X$ (for some fixed set X with $|X| > 2$) under composition of functions.

3. Show that the symmetric groups S_1 and S_2 of degrees 1 and 2 (respectively) are abelian.

4. Let $(i_1\ i_2\ \ldots\ i_r)$ be an arbitrary r-cycle in the symmetric group S_n of degree n. Determine the inverse of this permutation.

5. Consider the following permutations:

   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 4 & 6 & 3 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 7 & 8 & 3 & 4 & 1 \end{pmatrix}$$

   (a) Write σ and τ as products of disjoint cycles.

   (b) Calculate the products $\sigma\tau$, $\tau\sigma$ and τ^{-1}. Express your answers as products of disjoint cycles.

6. Write the following permutations as products of disjoint cycles:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 7 & 5 & 4 & 2 \end{pmatrix}$;

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 2 & 4 & 6 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 3 & 7 & 5 & 6 \end{pmatrix}$;

(c) $(1\ 2\ 3)\,(2\ 3\ 4)$;

(d) $(1\ 2)\,(2\ 3)\,(1\ 2)\,(2\ 3)$;

(e) $(1\ 3\ 5\ 2)\,\big((2\ 4\ 7)\,(1\ 6\ 2)\big)^{-1}$;

(f) $(2\ 4)\,(1\ 3\ 6)\,(3\ 7)\,\big((1\ 5)\,(2\ 6\ 4)\big)^{-1}$.

7. Consider the following permutations from the symmetric group of degree 10 given in two-row notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 7 & 5 & 9 & 10 & 1 & 4 & 8 & 2 & 6 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 10 & 3 & 8 & 1 & 2 & 9 & 7 & 6 & 4 \end{pmatrix}$$

(a) Write σ and τ as products of disjoint cycles.

(b) Calculate the products $\sigma^{-1}\tau\sigma$ and $\tau^{-1}\sigma\tau$. Give your answers as products of disjoint cycles.

8. Two permutations σ and τ in the symmetric group S_n are said to be *conjugate* in this group if there exists some $\rho \in S_n$ such that $\rho^{-1}\sigma\rho = \tau$. [This concept will be used significantly in a later chapter of the notes.]

(a) Let $\sigma = (i_1\ i_2\ \dots\ i_r)$ be an r-cycle in S_n and $\rho \in S_n$. Show that $\rho^{-1}\sigma\rho$ is the r-cycle

$$(i_1\rho\ \ i_2\rho\ \dots\ i_r\rho).$$

[Hint: Consider how the product $\rho^{-1}\sigma\rho$ moves points of the form $i_j\rho$ and how it moves points not of this form.]

(b) Find $\rho^{-1}\,(1\ 5\ 2\ 4)\,\rho$ where $\rho = (1\ 3)(2\ 5)$.

(c) Show that two permutations are conjugate in S_n if and only if they have the same disjoint cycle structure (that is, they have the same number of cycles of any length in their decomposition into disjoint cycles).
[Hint: Compare $\rho^{-1}\sigma_1\sigma_2\dots\sigma_k\rho$ and $(\rho^{-1}\sigma_1\rho)(\rho^{-1}\sigma_2\rho)\dots(\rho^{-1}\sigma_k\rho)$.]

(d) Find a permutation $\rho \in S_5$ such that $\rho^{-1}\,(1\ 2)\,(3\ 4\ 5)\,\rho = (3\ 4)\,(1\ 5\ 2)$.

9. For each of the symmetric groups S_4, S_5, S_6 and S_7, determine how many elements contain a 4-cycle when expressed as a product of disjoint cycles.