# School of Mathematics and Statistics

# MT2505 Abstract Algebra

# Problem Sheet V: Groups

1. For each of the following sets and proposed binary operation, determine whether or not this defines a group. Provide either proofs or an explanation of why this does not define a group. [You should not assume that the proposed binary operation is indeed a binary operation. Verifying that it is indeed a binary operation would be one step in a proof.]

   (a) The set $\{ 2^k \mid k \in \mathbb{Z} \}$ under multiplication of real numbers.

   (b) The set $\{ 3^k \mid k \in \mathbb{Z} \}$ under addition of real numbers.

   (c) The set $\{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \text{ with } a \neq 0 \text{ or } b \neq 0 \}$ under multiplication of real numbers.

   (d) The set $\{0, 1\}$ under multiplication of real numbers.

   (e) The set $\{ x \in \mathbb{R} \mid -1 \leqslant x \leqslant 1,\ x \neq 0 \}$ under multiplication of real numbers.

   (f) The set $\{ x \in \mathbb{Z} \mid x \equiv 1 \pmod 3 \}$ under addition.

   (g) The set
   $$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ a \neq 0 \right\}$$
   under matrix multiplication.

   (h) The set $\mathscr{P}(X)$ of all subsets of a non-empty set $X$ under the operation $\cap$ of intersection.

   (i) An arbitrary set $A$ with $|A| > 1$ under the operation $*$ defined by $x * y = x$ for all $x, y \in A$.

   (j) The set $\mathbb{R}$ of real numbers with operation $\bullet$ defined by $x \bullet y = x + y - xy$.

2. Consider the following sets (of congruence classes) under the given multiplication. Which of them form groups under this multiplication? Justify your answer.

   (a) $\mathbb{Z}/11\mathbb{Z}$ under multiplication modulo 11.

   (b) $(\mathbb{Z}/11\mathbb{Z}) \setminus \{0\}$, the set of non-zero congruence classes, under multiplication modulo 11.

   (c) $(\mathbb{Z}/20\mathbb{Z}) \setminus \{0\}$, the set of non-zero congruence classes, under multiplication modulo 20.

3. Consider the following sets (of congruence classes) under the given multiplication. Which of them form groups under this multiplication? Justify your answer.

   (a) $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.

   (b) $\{1, 2, 4, 5, 7, 8\}$ under multiplication modulo 9.

   (c) $\{1, 9, 11\}$ under multiplication modulo 14.

   (d) $\{1, 5, 9, 13\}$ under multiplication modulo 16.

4. Let $m$ be an integer with $m > 1$ and let $U_m$ denote the set of congruence classes in $\mathbb{Z}/m\mathbb{Z}$ consisting of elements coprime to $m$:

$$U_m = \{ x \in \mathbb{Z}/m\mathbb{Z} \mid \gcd(x, m) = 1 \}$$

Show that $U_m$ is a group under multiplication modulo $m$.

5. Let $G$ be a group.

   (a) Suppose that $x$ and $y$ are elements of $G$ such that $xy = x$. Show that $y = 1$, the identity element of $G$.

   (b) Suppose that $x$ and $y$ are elements of $G$ such that $xy = 1$. Show that $y = x^{-1}$.

6. Let $G$ be a group such that $x^2 = 1$ for all $x \in G$ (that is, every element squares to the identity element). Show that $G$ is abelian.

   [Hint: Consider $(xy)^2$.]

   Let $G = \{1, a, b, c\}$ be a group of order 4 such that $x^2 = 1$ for all $x \in G$. Construct the Cayley table of $G$. Which group already met in lectures has the same pattern of entries in its Cayley table?

   Can you construct other examples of groups $G$, with different orders, that satisfy the condition that $x^2 = 1$ for all $x \in G$?