MRQ 2024

School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet IV: Congruences and Modular Arithmetic

1. Let m be an integer with $m > 1$. Suppose that a and b are integers with $a \equiv b \pmod{m}$. Show that
$$a^n \equiv b^n \pmod{m}$$
for all natural numbers n.

2. Using congruence modulo 10, determine the last digit of the following numbers (when expressed in the usual base 10 notation):
$$8^2, \quad 8^3, \quad 8^4, \quad 8^5, \quad 8^{1000}, \quad 2^{82,589,933} - 1$$
(The last is the largest known prime currently known prime according to https://primes.utm.edu/largest.html.)

3. Is $\mathbb{Z}/20\mathbb{Z}$ a field under the operations of modular arithmetic? Justify your answer.

4. (a) Find the multiplicative inverses of the elements 4, 5 and 6 in $\mathbb{F}_{11}$.
   (b) Solve the equation $4x \equiv 9 \pmod{11}$.

5. Let m be a positive integer with $m > 1$. Show that $a \in \mathbb{Z}/m\mathbb{Z}$ has a multiplicative inverse if and only if $\gcd(a, m) = 1$.

   Deduce that if a and b are positive integers with $a, b > 1$, then a has a multiplicative inverse modulo b if and only if b has a multiplicative inverse modulo a.

   [Hint: If c is a multiplicative inverse for a modulo m, interpret what $ac \equiv 1 \pmod{m}$ means and consider what you can conclude about common divisors of a and m.]

6. Find the multiplicative inverse of 25 modulo 77 (that is, the inverse of 25 as an element of the ring $\mathbb{Z}/77\mathbb{Z}$).

   Solve the equation $25x \equiv 7 \pmod{77}$.

7. (a) Determine which elements of $\mathbb{Z}/9\mathbb{Z}$ have a multiplicative inverse and calculate their inverses.
   (b) Find all solutions in $\mathbb{Z}/9\mathbb{Z}$ of the following equations or show that no solutions exist:

   (i) $5x - 1 \equiv 0 \pmod{9}$, (ii) $3x + 7 \equiv 0 \pmod{9}$, (iii) $x(x+3) \equiv 0 \pmod{9}$