School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet III: Equivalence Relations

1. The relation $<$ ("less than") on the integers $\mathbb{Z}$ is not an equivalence relation. Which of the three conditions (reflexive, symmetric, and transitive) does this relation satisfy and which does it not satisfy?

   [Justify your answer by providing proofs for those that it satisfies and examples to demonstrate which conditions it does not satisfy.]

2. Define a relation $\sim$ on $\mathbb{R}^2$ by

$$(x_1, y_1) \sim (x_2, y_2) \qquad \text{when } |x_1| + |y_1| = |x_2| + |y_2|.$$

   Show that this relation $\sim$ is an equivalence relation on $\mathbb{R}^2$.

   If $(x, y) \in \mathbb{R}^2$, describe the equivalence class of the point (x, y) geometrically.

3. Let A and B be sets and let $f \colon A \to B$ be any function. Define a relation $\sim$ on the set A by the rule

$$a \sim b \qquad \text{when } f(a) = f(b)$$

   for elements $a, b \in A$.

   Show that $\sim$ is an equivalence relation on A.

   Take the specific example of $A = B = \mathbb{R}$ and the function f defined by $f(x) = x^2$. For the equivalence relation $\sim$ defined as above, what are the equivalence classes?

4. Let A be a set and $\mathcal{P} = \{ B_i \mid i \in I \}$ be a partition of A. Define a relation $\sim$ on A by $a \sim b$ if and only if a and b lie in the same part B_i.

   (a) Show that $\sim$ is an equivalence relation on A.

   (b) If $a \in A$, show that $[a] = B_i$, where B_i is the part to which a belongs.

5. Let A be a set and suppose that $\sim_1$ and $\sim_2$ are equivalence relations on A. Define a new relation $\sim$ on A by the rule

$$a \sim b \qquad \text{when both } a \sim_1 b \text{ and } a \sim_2 b.$$

   Show that $\sim$ is also an equivalence relation on A.

   If $a \in A$, how is the equivalence class of a under the relation $\sim$ determined by the equivalence classes of the same point using the relations $\sim_1$ and $\sim_2$?