School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet X: Homomorphisms, normal subgroups, and quotient groups

1. Which of the following functions ϕ are homomorphisms? In each case, justify your answer.
   (a) The function ϕ from the additive group $\mathbb{R}$ of real numbers to itself given by $x\phi = x/2$.
   (b) The function ϕ from the additive group $\mathbb{Z}$ of integers to itself given by $x\phi = x^2 - x$.
   (c) The function ϕ from the multiplicative group $\mathbb{R}^*$ of non-zero real numbers to the additive group $\mathbb{R}$ of real numbers given by $x\phi = \log|x|$.

2. Let G be any group and fix $g \in G$. Define the function $\tau \colon G \to G$ by $x\tau = g^{-1}xg$.
   (a) Show that τ is a homomorphism.
   (b) Show that τ is bijective.

3. Let G be the set of matrices of the form
$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}.$$
   (a) Show that G is a group under matrix multiplication.
   (b) Determine whether or not G is an abelian group.
   (c) Show that the function $\phi \colon G \to \mathbb{Z}$ given by
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \phi = a + c$$
   is a homomorphism from G to the additive group $\mathbb{Z}$ of integers.
   (d) Determine the kernel of ϕ. Is this kernel abelian?

4. Consider the groups of congruence classes with multiplicative inverses modulo 8 and modulo 16:
$$U_8 = \{1, 3, 5, 7\}$$
$$U_{16} = \{1, 3, 5, 7, 9, 11, 13, 15\}$$
   Consider the function $\phi \colon U_{16} \to U_8$ given by
$$x\phi = x \pmod 8$$
   (that is, $1\phi = 1$, …, $7\phi = 7$, $9\phi = 1$, $11\phi = 3$, …).
   Show that ϕ is a homomorphism. Is ϕ surjective? Determine the kernel of ϕ.
   Construct an injective homomorphism $\psi \colon U_8 \to U_{16}$.

5. Let G and H be groups and let $\phi\colon G \to H$ be a surjective homomorphism.

   (a) Show that if G is abelian, then H is also abelian.

   (b) Show that if G is cyclic, then H is also cyclic.

   (c) If H is abelian, is it necessarily the case that G is also abelian? Give a proof or a counterexample as appropriate.

   (d) If H is cyclic, is it necessarily the case that G is also cyclic? Give a proof or a counterexample as appropriate.

6. (a) Let $H = \langle (1\ 2\ 3\ 4\ 5\ 6) \rangle$ be the cyclic subgroup of S_6 generated by the given 6-cycle. Show that H is isomorphic to the additive group $\mathbb{Z}/6\mathbb{Z}$ of congruence classes modulo 6.

   (b) Let U_{14} denote the group of congruence classes that have a multiplicative inverse modulo 14.

      i. List the elements of U_{14}.

      ii. Show that U_{14} is a cyclic group.

      iii. Is U_{14} isomorphic to the group H in part (a)?

   (c) Let U_{16} denote the group of congruence classes that have a multiplicative inverse modulo 16.

      i. Show that U_{16} is not a cyclic group.

      ii. Is U_{16} isomorphic to the group H in part (a)?

   (d) Let $\mathbb{R}^+ = \{\, x \in \mathbb{R} \mid x > 0 \,\}$. Show that the function $f\colon \mathbb{R} \to \mathbb{R}^+$ given by $xf = \mathrm{e}^x$ is a bijection.
   [You may use the fact that e^x is a strictly increasing function of x.]

   (e) Show that the additive group $\mathbb{R}$ of real numbers is isomorphic to the set $\mathbb{R}^+$ of positive real numbers viewed as a group under multiplication.

7. For each of the following pairs of groups (as defined in the course) determine whether or not they are isomorphic:

   (a) the multiplicative group $\mathbb{F}_7^*$ and the additive group $\mathbb{Z}/6\mathbb{Z}$;

   (b) the group of isometries of a (non-square) rectangle and the Klein 4-group V_4;

   (c) the symmetric group S_3 and the additive group $\mathbb{Z}/6\mathbb{Z}$;

   (d) the symmetric group S_3 and the dihedral group D_6;

   (e) the alternating group A_4 and the dihedral group D_{12}.

8. Let G be a group. For each of the following statements, either prove the statement or provide a counterexample showing that in general the statement is not true:

   (a) If M and N are normal subgroups of G, then the intersection $M \cap N$ is a normal subgroup of G.

   (b) If M and N are normal subgroups of G, then the union $M \cup N$ is a normal subgroup of G.

   (c) If M and N are subgroups of G such that N is a normal subgroup of G and M is a normal subgroup of N, then M is a normal subgroup of G.

9. Let G be a group and N be a normal subgroup. Define a function $\pi\colon G \to G/N$ from G to the quotient group G/N by

$$\pi\colon x \mapsto Nx.$$

Show that π is a homomorphism.

Is π surjective? Determine the kernel of π.

[This homomorphism is called the *natural homomorphism* from G to the quotient group.]

10. Let G be a group and N be a subgroup of G of index 2. Show that N is a normal subgroup of G.

[Hint: What are the left cosets and right cosets of N in G?]

11. Let G be an abelian group.

(a) Show that every subgroup of G is normal in G.

(b) Let F be the set of elements of G of finite order. Show that F is a subgroup of G.

(c) Show that every non-identity element of the quotient group G/F has infinite order.

12. Consider the group

$$U_{16} = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

of congruence classes modulo 16 with multiplicative inverses. The subgroup $H = \langle 15 \rangle$ is a normal subgroup since U_{16} is abelian.

(a) Calculate the cosets of H in U_{16}.

(b) Determine the Cayley table of the quotient group U_{16}/H.