School of Mathematics and Statistics

MT2505 Abstract Algebra

Problem Sheet I: Binary Operations; Rings and Fields

1. Let $\mathbb{Z}[\sqrt{2}] = \{\, a + b\sqrt{2} \mid a, b \in \mathbb{Z} \,\}$ be the set of all real numbers of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

   Show that $\mathbb{Z}[\sqrt{2}]$ is a ring under the usual addition and multiplication of real numbers.

   [Don't forget to check that addition and multiplication is a binary operation on $\mathbb{Z}[\sqrt{2}]$. Checking the conditions to be a ring should then be very quick!]

2. Let $R = \{\, a/b \mid a, b \in \mathbb{Z},\ b \text{ is odd} \,\}$, the set of rational numbers that can be expressed with odd denominator. Show that R is a ring under the usual addition and multiplication of rational numbers.

3. Let $P = \{\, x \in \mathbb{Z} \mid x \geq 0 \,\}$, the set of non-negative integers. Is P a ring under the usual addition and multiplication of integers?

4. (a) Let

   $$R_1 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, b, d \in \mathbb{R} \right\},$$

   the set of 2×2 real matrices with $(2, 1)$-entry equal to 0. Show that R_1 is a ring under the usual addition and multiplication of matrices.

   (b) Let

   $$R_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}.$$

   Is R_2 a ring under the usual addition and multiplication of matrices?

5. Let n be a positive integer with $n > 2$. Show that the matrix ring $\mathrm{M}_n(\mathbb{Z})$ is *not* a commutative ring.

   [We handled $n = 2$ in lectures.]

6. Let X be any set and define $\mathscr{P}(X)$ to be the set of all subsets of X (including the empty set as one of these subsets). We call $\mathscr{P}(X)$ the *power set* of X.

   Define the following operations on $\mathscr{P}(X)$:

   $$A + B = (A \cup B) \setminus (A \cap B) \qquad \text{and} \qquad A \cdot B = A \cap B$$

   for subsets $A, B \subseteq X$. Show that $\mathscr{P}(X)$ is a ring under this definition of addition and multiplication.

   [Hint: To verify the ring axioms, it might help to draw Venn diagrams.]

7. Let $\mathbb{Q}[\sqrt{2}] = \{\, a + b\sqrt{2} \mid a, b \in \mathbb{Q} \,\}$ be the set of all real numbers of the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

   Show that $\mathbb{Q}[\sqrt{2}]$ is a field under the usual addition and multiplication of real numbers.

   [Hint: First show it is a ring. When you come to find multiplicative inverse, you wish to calculate $1/(a + b\sqrt{2})$ when $a, b \in \mathbb{Q}$. It might help you to remember what you do when dividing by complex numbers.]

8. **Definition:** Let $*$ be a binary operation on a set A. A *left identity* for $*$ is an element $e \in A$ such that $e * a = a$ for all $a \in A$. A *right identity* for $*$ is an element $f \in A$ such that $a * f = a$ for all $a \in A$.

   Consequently, an identity for $*$ (as given in Definition 1.2) is an element that is *both* a left identity and a right identity.

   (a) Let A be any set and define a binary operation $*$ on A by

   $$a * b = a \qquad \text{for all } a, b \in A.$$

   Show that $*$ is associative.
   Show that every element of A is a right identity for $*$.
   If $|A| > 1$, show that no element of A is a left identity.

   (b) Let A be any set and let $*$ be any binary operation on A. Suppose that e be a left identity for $*$ and f be a right identity for $*$. Show that $e = f$. [Hint: Consider $e * f$.]
   Deduce that e is then an identity (i.e., two-sided identity) for $*$.