SH/AN/MRQ/CMRD 2009

School of Mathematics and Statistics

MT2001 Mathematics: Linear Algebra

Problem Sheet I: Determinants & Inverses

1. What is wrong with the following?

$$\det\begin{pmatrix}2&1&3\\1&2&4\\2&1&1\end{pmatrix} = \det\begin{pmatrix}2&1&3\\0&3&5\\0&0&-2\end{pmatrix} \qquad \begin{matrix}r_2 \mapsto 2r_2 - r_1\\ r_3 \mapsto r_3 - r_1\end{matrix}$$
$$= -12.$$

What is the true value of the determinant?

2. Given that

$$A = \begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} \qquad \text{where} \quad \det A = 5,$$

determine the determinants of the following matrices

$$\begin{pmatrix}d&e&f\\g&h&i\\a&b&c\end{pmatrix}, \qquad \begin{pmatrix}-a&-b&-c\\2d&2e&2f\\g&h&i\end{pmatrix}, \qquad \begin{pmatrix}a+d&b+e&c+f\\d&e&f\\g&h&i\end{pmatrix}.$$

3. Using row operations and the properties of determinants, show that

$$\det\begin{pmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{pmatrix} = (b-a)(c-a)(c-b).$$

4. By applying row operations, obtain a factorisation of the determinant on the left-hand side and hence solve the equation

$$\det\begin{pmatrix}x&a&a&a\\a&x&a&a\\a&a&x&a\\a&a&a&x\end{pmatrix} = 0.$$

5. Calculate the adjugate matrix and hence the inverse of each of the following two matrices:

$$\text{(i)} \quad \begin{pmatrix}1&2&3\\2&1&1\\3&1&1\end{pmatrix}, \qquad \text{(ii)} \quad \begin{pmatrix}1&1&3\\4&1&1\\1&1&2\end{pmatrix}.$$

6. Prove that for 2×2 matrices:

   (a) If one column is a scalar multiple of another then the determinant is 0.

   (b) $\det(AB) = \det A \cdot \det B$.

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Problem Sheet II: Vector Spaces

1. Which of the following are vector spaces (with addition and scalar multiplication defined in the usual way in each case)?

   (a) The set of 2×2 matrices of the form
   $$\begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix}$$
   with a a real number.

   (b) The set of 2×2 matrices of the form
   $$\begin{pmatrix} a & a^2 \\ a^3 & a^4 \end{pmatrix}$$
   with a a real number.

   (c) The set of real integrable functions defined on $[0, 1]$ such that
   $$\int_0^1 f(x)\,\mathrm{d}x = 0.$$

   (d) The set of real integrable functions defined on $[0, 1]$ such that
   $$\int_0^1 f(x)\,\mathrm{d}x = 1.$$

   (e) The set of column vectors of real numbers $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying
   $$3x + 4y = 0.$$

2. For the given vector space V and subset $W \subseteq V$, in which of the following cases is W a subset of V?

   (a) $V = \mathbb{R}^3$, W the set of vectors of the form $\begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$;

   (b) $V = \mathbb{R}^3$, W the set of vectors of the form $\begin{pmatrix} a \\ 0 \\ 4a \end{pmatrix}$;

(c) $V = \mathcal{P}_2$, the set of polynomials of degree at most 2, W the set of polynomials of the form $ax^2 + bx$;

(d) $V = M_{n\times n}(\mathbb{R})$, the set of $n \times n$ real matrices, W the set of symmetric matrices. (A symmetric matrix is one that is equal to its transpose.)

3. Let V be the vector space of 2×2 real matrices with the usual operations for addition and scalar multiplication. Show that the set of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a subspace of V.

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MT2001 Mathematics: Linear Algebra

Problem Sheet III: Linear Independence and Bases

1. Which of the following sets of vectors are linearly independent and what is the dimension of the space that they span?

   (a)
   $$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

   (b) $\{x, 1 + x^2, 1 + x\} \subseteq \mathcal{P}_2$, the space of all polynomials of degree at most 2.

   (c) $\{\mathrm{e}^x, \mathrm{e}^{-x}, \cosh x, \sinh x\}$ inside the space of continuous functions on $[0, 1]$.

2. Show that
   $$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right\}$$
   is a basis for the space of 2×2 real matrices of the form
   $$\begin{pmatrix} a & b \\ a+b & a-b \end{pmatrix}.$$
   What is the dimension of this space?

3. Show that the set
   $$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$
   spans the space consisting of vectors of the form
   $$\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$$
   for a and b real numbers and express a typical vector in terms of this set.

   Is this set a basis for the space? What is the dimension of the space?

4. Find a basis for the set of vectors in $\mathbb{R}^4$ of the form
   $$\begin{pmatrix} a+b+c \\ a-b-c \\ b+c \\ 2a+b+c \end{pmatrix}.$$
   What is the dimension of the space? Express a typical vector from the space in terms of your chosen basis.

SH/AN/MRQ/CMRD 2009

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MT2001 Mathematics: Linear Algebra

Problem Sheet IV: Linear Transformations

1. Show that the following are linear maps.

   (a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ b+c \\ c+a \end{pmatrix}.$$

   (b) $T\colon \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ defined by
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & a+c \end{pmatrix}.$$

2. If T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ with
$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$
   what is $T\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

   What is the value of $T\begin{pmatrix} a \\ b \end{pmatrix}$ where a and b are any real numbers?

3. The mapping f from $\mathbb{R}^2$ to $\mathbb{R}^3$ has the property that
$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \qquad f\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}.$$
   Show that this is *not* a linear mapping.

4. What is the matrix of T, with respect to the standard bases, for each of the following linear transformations?

   (a) $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \\ b-c \\ b+c \end{pmatrix};$$

(b) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix};$$

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-b \\ b-c \\ c-a \end{pmatrix}.$$

In each of the above determine the rank of the linear transformation.

5. The linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -5x+13y \\ -7x+16y \end{pmatrix}.$$

Using the bases $\mathscr{B} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right\}$ for $\mathbb{R}^2$ and $\mathscr{B}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ for $\mathbb{R}^3$, show that the matrix $\mathrm{Mat}_{\mathscr{B},\mathscr{B}'}(T)$ for this mapping with respect to $\mathscr{B}$ and $\mathscr{B}'$ is

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{pmatrix}.$$

6. Find a basis for the row-space and a basis for the column-space of the following matrix over $\mathbb{R}$:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Hence determine the row-rank and the column-rank of this matrix and verify that they are equal.

7. Let $V = M_{2\times 2}(\mathbb{R})$, the space of 2×2 real matrices. We use the following basis for V:

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a typical 2×2 matrix has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = av_1 + bv_2 + cv_3 + dv_4;$$

i.e., the matrix has coordinates (a, b, c, d) in terms of this basis.

Find the matrix of the linear transformation $T\colon V \to V$, in terms of this standard basis, defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a. \end{pmatrix}$$

Find a basis for the image $T(V)$.

8. Let V be the subspace of the space of real functions spanned by $\{\sin, \cos\}$, i.e., all functions $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a\sin x + b\cos x$.

   Let $T\colon V \to \mathbb{R}$ be given by

$$T(f) = \int_0^\pi f(x)\,\mathrm{d}x.$$

   Show that T is a linear transformation. Determine the kernel of T and hence find the nullity of T.

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Problem Sheet V: Eigenvalues and Eigenvectors

1. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}.$$

2. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ -4 & 2 & 0 \\ 6 & 0 & 0 \end{pmatrix}.$$

3. Let A be a matrix in $M_{n\times n}(\mathbb{C})$ and let λ be an eigenvalue of A. By introducing an eigenvector, show that λ^2 is an eigenvalue of A^2. Generalise your argument to show that λ^t is an eigenvalue of A^t for all $t \in \mathbb{N}$.

4. Show that the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   is $(x-1)(x^3+2x^2-x-2)$.

   Hence find the eigenvalues and the associated eigenvectors. Is the matrix diagonalisable? If so, determine the matrix P such that $P^{-1}AP$ is a diagonal matrix.

5. Find an orthogonal matrix which diagonalises

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

6. Find a unitary matrix which diagonalises

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}.$$