MT2001 Mathematics: Linear Algebra

MRQ

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Introduction

Linear algebra arises out of the study of matrices and vectors. In MT1002 we have seen how matrices arise naturally in two settings, first in the context of solving systems of linear equations and second as specifying certain geometric transformations. Vectors also arise naturally in the geometric setting of 2-dimensional and 3-dimensional real space. In this part of the course, we shall see how the concept of a vector space has been generalised from its geometric origins to an algebraic object which has applications in a range of different settings:

- pure mathematics (e.g., geometry, algebra, functional analysis, etc.),
- applied mathematics (e.g., spaces of solutions to differential equations, etc.),
- physics (e.g., quantum mechanics, etc.),
- etc.

A theme that arises as one studies linear algebra is that linearity makes solving many problems much easier. The sort of information that we are able to determine concerning transformations that are linear is far beyond what we can hope to achieve for arbitrary functions. The second theme that can be observed is that vector spaces form the natural setting in which to study linearity.

The principal methods that will be covered in this course are the following:

- how to calculate the determinant and inverse of a matrix;
- how to recognise a vector space;
- how to show a subset of a vector space is linearly independent;
- how to recognise a linear transformation;
- how to calculate the rank and nullity of a linear transformation;
- how to find eigenvalues and eigenvectors and hence diagonalise a linear transformation.

All these ideas are taken further in the module MT3501 Linear Mathematics. This set of notes have been written so that the notations used in both modules are consistent. The following textbooks are relevant to both modules:

- T. S. Blyth & E. F. Robertson, *Basic Linear Algebra, Second Edition*, Springer Undergraduate Mathematics Series (Springer-Verlag 2002);
- R. Kaye & R. Wilson, *Linear Algebra*, Oxford Science Publications (OUP 1998).

Chapter 1

Determinants and Inverses

Recall from MT1002 that given a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1.1}$$

we associate an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and a column vector

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then solving the system of equations (1.1) is equivalent to solving the equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

for the column vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In this section, we shall consider matrices in more detail, returning to the topic of solving systems of equations as above at the end.

An $m \times n$ matrix A is a rectangular array of entries with m rows and n columns. But what are these entries? These entries come from some *field* F. We shall not give a detailed definition of such a mathematical object, but the following is sufficient for our needs.

Definition 1.1 A *field* is a structure in which we can add, multiply and subtract any two elements and we can divide any element by a *non-zero* element. Furthermore, all natural rules of arithmetic should hold.

We refer to the elements in our field as *scalars*.

A complete and precise definition of what is meant by a field will be given in MT3501 (and also in MT4517 Rings and Fields). For this course, it is enough to say that the two standard examples of fields are the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. These will be the only fields that we shall use here.

Let $A = [a_{ij}]$ be a square $n \times n$ matrix with entries from a field F. This notation indicates that a_{ij} denotes the entry in the ith row and jth column of A. The determinant of A was defined in MT1002 in an inductive manner. We recall this.

First associate to each position in the $n \times n$ matrix a sign from the following grid:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Definition 1.2 Fix a particular entry in the $n \times n$ square matrix A. The *cofactor* of this entry is defined to be:

> the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the row and column containing this entry $\times$ the sign associated with this entry.

The determinant of the $n \times n$ matrix $A = [a_{ij}]$ is then obtained by multiplying each entry in the top row by its cofactor and adding the resulting products. We denote this determinant by $\det A$ or $|A|$.

Thus to calculate a determinant of a 3×3 matrix, we first need to calculate that of a 2×2 matrix, and so on.

For a 1×1 matrixx, $A = (a_{11})$, the determinant is given by

$$\det A = \det(a_{11}) = a_{11}.$$

For a 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the determinant is

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$
$$= a_{11}a_{22} - a_{12}a_{21}.$$

For a 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the determinant is

$$\begin{aligned}\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \\ &\qquad + a_{13}\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\qquad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

Properties of determinants

We shall list various standard properties of determinants. Many of these are already known from MT1002 and all are very easily checked by direct calculation for determinants of 2×2 and 3×3 matrices. The proofs for arbitrary matrices are more complicated and are omitted.

Definition 1.3 If $A = [a_{ij}]$ is an $m \times n$ matrix, the *transpose* A^{T} of A is $n \times m$ matrix whose (i, j)th entry is a_{ji}.

So A^{T} is obtained from A by interchanging rows and columns.

Proposition 1.4 *If A is a square matrix, then*

$$\det A^{\mathrm{T}} = \det A.$$

Consequently, any property of determinants that is expressed in terms of rows also holds when expressed in terms of columns.

Theorem 1.5 *The determinant can be expanded in terms of any row or column: just go along the particular row or column, multiplying each entry by its cofactor and then add.*

Example 1.6 *Calculate the determinant of*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Solution: An expansion in terms of the top row gives:

$$\begin{aligned} \det A &= 1 \times \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 0 - 2 \times (-3) + 3 \\ &= 0 + 6 + 3 = 9. \end{aligned}$$

An equally valid solution would be to expand in terms of the middle row:

$$\begin{aligned} \det A &= -1 \times \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + (-1) \times \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ &= (-1) \times (-3) + (-1) \times (-3) - 1 \times (-3) \\ &= 3 + 3 + 3 = 9. \end{aligned}$$

Corollary 1.7 *If any row of a square matrix is zero, then the determinant is zero.*

PROOF: Expand along the zero row; all terms in the sum are zero, so the determinant is zero. □

Theorem 1.8 *Let A be a square matrix and let r_i denote the ith row of A.*

(i) *If row r_i is multiplied by the scalar α, then the determinant is also multiplied by α. Consequently, a common factor in any row can be taken outside the determinant.*

(ii) *If row r_i is replaced by $r_i + \alpha r_j$, then the determinant is unchanged. Consequently, if two rows are equal, or even proportional to each other, the determinant is equal to zero.*

Example 1.9 *Calculate the determinant of*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Solution: This is an alternative solution to Example 1.6, but this time we shall apply row operations:

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & -3 & -3 \end{vmatrix} \qquad \begin{matrix} r_2 \mapsto r_2 - r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{matrix}$$

$$= \begin{vmatrix} -3 & 0 \\ -3 & -3 \end{vmatrix} \qquad \text{(expanding along first column)}$$

$$= 9.$$

Proposition 1.10 *Interchanging two rows in a square matrix, changes the sign of the determinant.*

Theorem 1.11 (i) *If A and B are $n \times n$ matrices, then* $\det(AB) = \det A \cdot \det B$.

(ii) *The determinant of a diagonal matrix is the product of the entries on the diagonal:*

$$\det \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix} = \alpha_1 \alpha_2 \ldots \alpha_n.$$

(iii) $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

Inverses

Definition 1.12 Let A be a square matrix. We say that A is *invertible* with *inverse* A^{-1} if

$$AA^{-1} = A^{-1}A = I$$

(where I is the $n \times n$ identity matrix).

One method to find the inverse of a $n \times n$ matrix A was described in MT1002:

(i) Write A and I together in a single $n \times 2n$ matrix:

$$(A \mid I).$$

(ii) Apply row operationrs to convert the left hand matrix into the identity matrix:

$$(I \mid B).$$

(iii) The matrix now appearing on the right-hand side is the inverse of A:

$$A^{-1} = B.$$

In this section, we shall describe an alternative method for finding inverses. The first ingredient is the following:

Definition 1.13 Let A be a square matrix. The *adjugate* of A, denoted by $\operatorname{adj} A$, is constructed from A by the following two steps:

(i) Replace each entry in the matrix by its cofactor;

(ii) take the transpose of the resulting matrix.

Example 1.14 *Find the adjoint of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

and calculate the product $A \cdot \operatorname{adj} A$.

Solution: Recall that the distribution of the signs associated with the cofactors is:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

Each cofactor is the determinant of a 2×2 matrix adjusted by the above signs, so we calculate the matrix of cofactors is

$$\begin{pmatrix} +0 & -(-1) & +(-3) \\ -2 & +1 & -(-1) \\ +2 & -1 & +1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -3 \\ -2 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

Hence

$$\operatorname{adj} A = \begin{pmatrix} 0 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & 1 & 1 \end{pmatrix}.$$

Thus

$$A \cdot \operatorname{adj} A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I = (\det A)I.$$

We will now show that what was observed in the previous example is actually true in full generality.

Theorem 1.15 *If A is a square matrix, then*

$$A \cdot \operatorname{adj} A = (\det A)I.$$

PROOF: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Write

$$B = [b_{ij}] = \operatorname{adj} A$$

for the adjugate of A. Here b_{ij} is the cofactor of the (j, i)th entry of A.

Now consider the (i, i)th entry of the product AB. It equals

$$\sum_{k=1}^{n} a_{ik} b_{ki}.$$

This is the sum of each entry in the ith row multiplied by its cofactor (b_{ki} is the cofactor of the (i, k)th entry), so this sum is simply the determinant of A calculated by expanding along the ith row. Hence the diagonal entries in AB all equal $\det A$.

Now consider the $(1, 2)$ entry of the product AB. It equals

$$\sum_{k=1}^{n} a_{1k} b_{k2}.$$

This is much like expanding the determinant of A using the cofactors of the second row, but we are multiplying them by the entries of the first row instead of the second. Hence this sum equals the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

when we expand along the second row. This matrix has repeated rows, so its determinant is zero. Hence the $(1, 2)$ entry of AB equals 0. The same holds for all other off diagonal entries by the same argument. Hence

$$A \cdot \operatorname{adj} A = AB = (\det A) I.$$

□

This enables us to calculate the inverse of a square matrix directly from its adjugate:

Theorem 1.16 *A square matrix A possesses an inverse if and only if its determinant is non-zero. In this case,*

$$A^{-1} = \left(\frac{1}{\det A} \right) \operatorname{adj} A.$$

PROOF: If A has an inverse, then $AA^{-1} = I$ and, upon taking determinants, we find

$$(\det A)(\det A^{-1}) = \det I = 1$$

and therefore $\det A \neq 0$.

Conversely, if $\det A \neq 0$, then $B = (1/\det A) \operatorname{adj} A$ exists and we see

$$AB = \frac{1}{\det A} A \cdot \operatorname{adj} A = \frac{1}{\det A} \cdot (\det A) I = I$$

and hence $B = A^{-1}$. □

Example 1.17 *Find the inverse of*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

Solution: The determinant of A is

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -1 \\ 0 & 1 & 1 \end{vmatrix} = -6 + 1 = -5.$$

Taking cofactors and the transpose, we find that the adjugate of A is

$$\operatorname{adj} A = \begin{pmatrix} -6 & -1 & 4 \\ -4 & 1 & 1 \\ 9 & -1 & -6 \end{pmatrix}.$$

Hence

$$A^{-1} = \begin{pmatrix} \frac{6}{5} & \frac{1}{5} & -\frac{4}{5} \\ \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{9}{5} & \frac{1}{5} & \frac{6}{5} \end{pmatrix}.$$

Lemma 1.18 *If A and B are invertible $n \times n$ matrices, then AB is also invertible, with*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PROOF: We calculate

$$AB \cdot B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly $B^{-1}A^{-1} \cdot AB = I$. Hence AB has $B^{-1}A^{-1}$ as its inverse. □

Solving systems of linear equations

We shall now illustrate how the above concepts arise in the context of solving systems of simultaneous linear equations. For example, consider the system of equations

$$\begin{aligned} x + 3y + 3z &= 2 \\ x + 4y + 3z &= 3 \\ x + 3y + 4z &= 4. \end{aligned}$$

In matrix form, this is equivalent to

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix},$$

or

$$A\boldsymbol{x} = \boldsymbol{b},$$

where

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Now if the matrix A is invertible, we can multiply by its inverse and deduce that $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is the unique solution to our system of equations. We conclude:

Theorem 1.19 *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ if $\det A \neq 0$.*

Chapter 2

Vector Spaces

The concept of vectors in 2- and 3-dimensional real space should be familiar to students taking this course. In this chapter, we are going to generalise this concept in such a way to cover a broad range of mathematical system. This will include matrices, polynomials, functions, and so on.

Consider the set of vectors in 3-dimensional real space. What properties do these vectors have? The following are the properties which we shall require our generalisation to also satisfy:

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors, they have a sum $\boldsymbol{u} + \boldsymbol{v}$;
- if α is a scalar and $\boldsymbol{v}$ is a vector, then $\alpha\boldsymbol{v}$ is a vector;
- these addition and scalar multiplication operations satisfy natural-looking laws.

The following definition gives a mathematical object that fulfils our requirements. It is the principal object of interest for this part of the course.

Definition 2.1 Let F be a field of scalars (typically $\mathbb{R}$ and $\mathbb{C}$ for this course). A *vector space* over F is a set V together with two operations

$$\begin{aligned} V \times V &\to V \\ (u, v) &\mapsto u + v \end{aligned} \qquad\qquad \begin{aligned} F \times V &\to V \\ (\alpha, v) &\mapsto \alpha v, \end{aligned}$$

called *addition* and *scalar multiplication*, respectively, such that

(i) $u + v = v + u$ for all $u, v \in V$;

(ii) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$;

(iii) there exists a vector $\mathbf{0}$ in V such that $v + \mathbf{0} = v$ for all $v \in V$;

(iv) for each $v \in V$, there exists a vector $-v$ in V such that $v + (-v) = \mathbf{0}$;

(v) $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in V$ and all scalars α;

(vi) $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in V$ and all scalars α, β;

(vii) $(\alpha\beta)v = \alpha(\beta v)$ for all $v \in V$ and all scalars α, β;

(viii) $1v = v$ for all $v \in V$.

It is always important to distinguish between the zero vector $\mathbf{0}$ in the vector space V and the zero scalar in our field F. Consequently, we shall use bold-face notation in these lecture notes (and an underlined zero on the white-board) to denote the zero vector.

Examples of vector spaces

As a first example of a vector space, we shall generalise the concept of vectors in 3-dimensional space to arbitrary dimensions.

Example 2.2 Let $\mathbb{R}^n$ denote the set of column vectors of length n:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In $\mathbb{R}^n$, we can add and multiply by scalars:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

The zero vector is

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and negatives are given by

$$-\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}.$$

With this information, it is very easy to check the eight axioms of a vector space. The only obstacle is the amount of paper it takes up, not the actual process of completing the checks!

Comment: We could instead work with row vectors $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$. In many ways, there is very little to distinguish between them and the principal difference is notational. For example, many applied mathematics modules will use row vectors for much of what they do. It is, on the whole, often more convenient to use row vectors than column vectors since they take up less space on the page. For the sake of the study of linear algebra, there is one principal difference and it is for this reason that we shall use column vectors. They will fit into our theory much better when we come to study linear transformations and, in particular, the matrix of a linear transformation.

We shall now give some further examples of vector spaces to illustrate how widely the concept is applicable. For the first, we shall check all eight axioms of a vector space, just to indicate how the process is completed. For those that follow, we shall omit most of the checking since it is usually straightforward.

Example 2.3 *Let $M_{m\times n}(F)$ be the set of $m \times n$ matrices with entries from a field F (as always, $F = \mathbb{R}$ or $\mathbb{C}$ in this course). Show that $M_{m\times n}(F)$ forms a vector space with respect to the usual addition and scalar multiplication.*

Solution: We are using the usual definition of addition of matrices, while scalar multiplication is to be where each entry of a matrix is multiplied by the scalar. Thus if $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices, we define

$$A + B = [a_{ij} + b_{ij}] \qquad \text{and} \qquad \alpha A = [\alpha a_{ij}].$$

First note that these both give us an $m \times n$ matrix, so we do have operations on $M_{m\times n}(F)$.

We now check the axioms of a vector space.

(i) $A + B = B + A$ holds for all matrices $A, B \in M_{m\times n}(F)$.

(ii) $(A+B)+C = A+(B+C)$ holds for all matrices $A, B, C \in M_{m\times n}(F)$.

(iii) Let $\mathbf{0}$ denote the matrix all of whose entries are zero. Then $A + \mathbf{0} = [a_{ij} + 0] = [a_{ij}] = A$ for all $A \in M_{m\times n}(F)$.

(iv) Given $A = [a_{ij}]$, take $-A = [-a_{ij}]$ be the matrix whose entries are the negatives of those in A. Then $A + (-A) = \mathbf{0}$.

(v) If $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$\begin{aligned} \alpha(A + B) = \alpha([a_{ij}] + [b_{ij}]) &= \alpha[a_{ij} + b_{ij}] = [\alpha(a_{ij} + b_{ij})] \\ &= [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B. \end{aligned}$$

(vi) A similar calculation shows $(\alpha+\beta)A = \alpha A + \beta A$ for $\alpha, \beta \in F$ and $A \in M_{m\times n}(F)$. (Supply the missing details!)

(vii) Equally $(\alpha\beta)A = \alpha(\beta A)$ for all $\alpha, \beta \in F$ and $A \in M_{m\times n}(F)$. (Supply the missing details!)

(viii) Finally $1A = 1[a_{ij}] = [1a_{ij}] = [a_{ij}] = A$ for all $A \in M_{m\times n}(F)$.

Hence the space of $m \times n$ matrices forms a vector space over F.

Example 2.4 *Let $\mathcal{P}_n$ be the set of polynomials*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

of degree at most n with coefficients from $\mathbb{R}$ with the usual addition (add the coefficients) and scalar multiplication (multiply each coefficient by the scalar). Show that $\mathcal{P}_n$ is a vector space over $\mathbb{R}$.

Sketch solution: Consider $f(x), g(x) \in \mathcal{P}_n$, say

$$f(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n \quad \text{and} \quad g(x) = b_0+b_1x+b_2x^2+\cdots+b_nx^n,$$

where all a_i and b_i are real coefficients. Then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \in \mathcal{P}_n$$

and, if $\alpha \in \mathbb{R}$,

$$\alpha f(x) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n \in \mathcal{P}_n.$$

The axioms (i)–(viii) are easily checked (in exactly the same way as was done in Example 2.3), the zero vector in $\mathcal{P}_n$ is

$$0 = 0 + 0x + 0x^2 + \cdots + 0x^n$$

while, for $f(x)$ as above,

$$-f(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n.$$

Example 2.5 Let $\mathcal{F}_{\mathbb{R}}$ denote the set of all real-valued functions of a real variable $f\colon \mathbb{R} \to \mathbb{R}$. This set forms a vector space over $\mathbb{R}$, where the addition and scalar multiplication are given by

$$(f + g)(x) = f(x) + g(x) \qquad (\alpha f)(x) = \alpha f(x).$$

The zero vector is the function f_0 given by

$$f_0(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Basic properties of vector spaces

Proposition 2.6 *Let V be a vector space over a field F. Let $v \in V$ and $\alpha \in F$. Then*

(i) $\alpha\mathbf{0} = \mathbf{0}$;

(ii) $0v = \mathbf{0}$;

(iii) *if* $\alpha v = \mathbf{0}$, *then either* $\alpha = 0$ *or* $v = \mathbf{0}$;

(iv) $(-\alpha)v = -\alpha v = \alpha(-v)$.

PROOF: (i) Let $w = \alpha\mathbf{0}$. Use condition (v) of Definition 2.1 and the fact that $\mathbf{0} + \mathbf{0} = \mathbf{0}$ by condition (iii) to give

$$w = \alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0} = w + w.$$

Now add $-w$ to both sides to yield

$$\mathbf{0} = w + (-w) = (w + w) + (-w) = w + (w + (-w)) = w + \mathbf{0} = w$$

(using conditions (ii) and (iv) from the definition).

(ii) Use condition (vi) of Definition 2.1 to give

$$0v = (0 + 0)v = 0v + 0v$$

and then add $-0v$ just as in part (i) to give $\mathbf{0} = 0v$.

(iii) Suppose that $\alpha v = \mathbf{0}$, but that $\alpha \neq 0$. Then we can multiply by $1/\alpha$ to give

$$\tfrac{1}{\alpha}(\alpha v) = \tfrac{1}{\alpha}\mathbf{0} = \mathbf{0} \qquad \text{(by part (i))}.$$

Now use conditions (viii) and (vii) of the definition to give

$$v = 1v = \left(\tfrac{1}{\alpha} \cdot \alpha\right) v = \tfrac{1}{\alpha}(\alpha v) = \mathbf{0}.$$

Hence if $\alpha v = \mathbf{0}$, either $\alpha = 0$ or $v = \mathbf{0}$.

(iv)

$$\alpha v + (-\alpha)v = (\alpha + (-\alpha))\, v = 0v = \mathbf{0},$$

so if we add $-\alpha v$ to both sides so as to cancel the first term on the left, we deduce

$$(-\alpha)v = -\alpha v.$$

Similarly,

$$\alpha v + \alpha(-v) = \alpha(v + (-v)) = \alpha\mathbf{0} = \mathbf{0}$$

and again adding $-\alpha v$, we deduce

$$\alpha(-v) = -\alpha v.$$

□

Subspaces

Definition 2.7 Let V be a vector space over a field F. A *subspace* W of V is a non-empty subset of V which itself forms a vector space under the same operations.

Hence, a subspace must obey all the axioms of its "parent" space. To check whether a set forms a subspace, we do not actually need to check every axiom. Most of them are immediately inherited from the fact that they hold in the parent space. Actually it is sufficient to ensure the closure of the operations within W. So to check whether W is a subspace of V, we need to check

- if $v, w \in W$, then $v + w \in W$;
- if $v \in W$ and α is any scalar, then $\alpha v \in W$.

From these two conditions, all other axioms follow. For example, setting $\alpha = 0$ in the second condition shows that

$$\mathbf{0} = 0v \in W.$$

(Here we use the fact that W is non-empty to find at least one vector v in W.) Hence W contains the zero vector of V and this ensures that axiom (iii) of a vector space is inherited by W.

Example 2.8 *Let W be the set of all vectors*

$$\begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

within the vector space $\mathbb{R}^3$. Show that W is a subspace of $\mathbb{R}^3$.

Solution: We need to show that W is non-empty and satisfies the above two conditions (i.e., is closed under addition and scalar multiplication). Taking $x = 0$ shows that

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W,$$

so W is non-empty. Now if v and w are vectors in W, say

$$v = \begin{pmatrix} x \\ x \\ x \end{pmatrix} \qquad \text{and} \qquad w = \begin{pmatrix} y \\ y \\ y \end{pmatrix},$$

then

$$v + w = \begin{pmatrix} x \\ x \\ x \end{pmatrix} + \begin{pmatrix} y \\ y \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix} \in W$$

and, for any scalar $\alpha \in \mathbb{R}$,

$$\alpha v = \alpha \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha x \\ \alpha x \end{pmatrix} \in W.$$

Hence W is a subspace of $\mathbb{R}^3$.

Example 2.9 *Let F be a field and $V = M_{2\times 2}(F)$ be the vector space of all 2×2 matrices with entries from F. Show that the set W of all diagonal matrices is a subspace of V.*

Solution: Here

$$W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in F \right\} \subseteq M_{2\times 2}(F).$$

This set is non-empty since, for example, it contains the zero matrix (take $a = b = 0$). Now let $A, B \in W$, say

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} \in W$$

and, if α is a scalar from F, then

$$\alpha A = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \in W.$$

Hence W is closed under addition and scalar multiplication, so we conclude W is a subspace of $V = M_{2\times 2}(F)$.

One might wonder whether all non-empty subsets of a vector space are actually subspaces, but it is easy to find examples of subsets which are not closed under addition or are not closed under scalar multiplication.

Example 2.10 *Let*

$$S = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

Show that S is not a subspace of $\mathbb{R}^3$.

Solution: We show that S is not closed under addition. For example,

$$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad w = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

both lie in S, but

$$v + w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \notin S.$$

Hence S is not a subspace of $\mathbb{R}^3$.

Example 2.11 *Consider the vector space $\mathcal{P}_3$ of all polynomials of degree at most 3 with real coefficients and let T be the subset of all polynomials of degree exactly 3. Show that T is not a subspace of $\mathcal{P}_3$.*

Solution: The set T contains the polynomials x^3 and $1 + x^2 - x^3$, but

$$x^3 + (1 + x^2 - x^3) = 1 + x^2 \notin T.$$

Hence T is not closed under addition and so is not a subspace.

Chapter 3

Linear Independence and Bases

Spanning sets

In a vector space we can multiply vectors by scalars and then add them. We shall investigate what happens if we apply such operations to some fixed set of vectors.

Throughout the following discussion we fix a vector space V over a field F and let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be some fixed set of vectors in V.

Definition 3.1 A vector v is a *linear combination* of the vectors in $\mathscr{A}$ if there are scalars α_1, α_2, …, α_k in F such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k.$$

The set of all linear combinations of the vectors in $\mathscr{A}$ is called the *span* of $\mathscr{A}$. We denote this by $\operatorname{Span}(\mathscr{A})$ or $\operatorname{Span}(v_1, v_2, \dots, v_k)$.

Theorem 3.2 *Let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in a vector space V (over a field F). Then the span of $\mathscr{A}$ is a subspace of V.*

As well as saying that W is the span of $\mathscr{A}$ in this situation, we shall also say that $\mathscr{A}$ is a *spanning set* for the space W.

PROOF: Write $W = \operatorname{Span}(\mathscr{A})$. First taking $\alpha_i = 0$ for all i, we see that W contains

$$0v_1 + 0v_2 + \dots + 0v_k = \mathbf{0}$$

(by Proposition 2.6(ii)). Hence at least W is non-empty.

Now let $v, w \in W$, so

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k \quad \text{and} \quad w = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_k v_k$$

for some scalars α_i and β_i. Hence

$$v + w = (\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \cdots + (\alpha_k + \beta_k)v_k \in W$$

and if α is any scalar in F then

$$\alpha v = (\alpha\alpha_1)v_1 + (\alpha\alpha_2)v_2 + \cdots + (\alpha\alpha_k)v_k \in W.$$

Hence W is a subspace of V. □

Example 3.3 *Define*

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *is a spanning set for* $\mathbb{R}^3$.

We refer to this as the *standard* spanning set.

Solution: A typical element in $\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has the form

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

for $x, y, z \in \mathbb{R}$. Hence $\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

Example 3.4 *Let*

$$\mathscr{A} = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}.$$

Determine the subspace of $M_{2\times 2}(F)$ *spanned by the set* $\mathscr{A}$.

Solution: A typical element in the space spanned by $\mathscr{A}$ has the form

$$\alpha\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \beta\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix}.$$

Hence $\mathrm{Span}(\mathscr{A})$ is the space of all diagonal 2×2 matrices.

Example 3.5 *Let* $V = \mathbb{R}^3$ *be the space of real vectors of length* 3. *Let*

$$\mathscr{A} = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Describe the subspace of V *spanned by* $\mathscr{A}$.

Solution: A typical vector in the space spanned by $\mathscr{A}$ has the form

$$\boldsymbol{v} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \beta \end{pmatrix}.$$

So a vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ lies in the subspace $\operatorname{Span}(\mathscr{A})$ if and only if $x = y$. Hence the subspace of V is the plane in $\mathbb{R}^3$ with equation $x - y = 0$.

Linear independence

Definition 3.6 Suppose that V is a vector space over a field F. A set $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ of vectors is called *linear independent* if the only solution to the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0}$$

(with $\alpha_i \in F$) is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

If $\mathscr{A}$ is not linearly independent, we shall call it *linearly dependent.*

Proposition 3.7 *Let $\mathscr{A}$ be a set of vectors in a vector space V.*

(i) *The set $\mathscr{A}$ is linearly independent if and only if no vector in the set can be expressed as a linear combination of the others.*

(ii) *The set $\mathscr{A}$ is linearly dependent if and only if some vector in the set can be expressed as a linear combination of the others.*

PROOF: The two statements are equivalent. We shall prove the second.

Suppose $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ is linear dependent. This means that there exist scalars α_1, α_2, …, α_k, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0}.$$

Let us suppose that it is α_j that is non-zero. Rearrange the previous equation to

$$\alpha_j v_j = -(\alpha_1 v_1 + \cdots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \cdots + \alpha_k v_k).$$

Therefore

$$v_j = \left(-\frac{\alpha_1}{\alpha_j}\right) v_1 + \cdots + \left(-\frac{\alpha_{j-1}}{\alpha_j}\right) v_{j-1} + \left(-\frac{\alpha_{j+1}}{\alpha_j}\right) v_{j+1} + \cdots + \left(-\frac{\alpha_k}{\alpha_j}\right) v_k.$$

Hence v_j is a linear combination of the other vectors v_1, …, v_{j-1}, v_{j+1}, …, v_k in $\mathscr{A}$.

Conversely, suppose one of the vectors in $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ is a linear combination of the others, say

$$v_j = \beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} + \cdots + \beta_k v_k.$$

Rearranging, we obtain

$$\beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1} + (-1) v_j + \beta_{j+1} v_{j+1} + \cdots + \beta_k v_k = \mathbf{0}.$$

This is an equation expressing the linear dependence of the vectors in $\mathscr{A}$ since not all coefficients are non-zero (the v_j has -1 as its coefficient). Hence $\mathscr{A}$ is linear dependent. □

Example 3.8 *Show that the vectors*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$ *are linearly independent.*

Solution: Consider the equation

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \mathbf{0}.$$

We need to show that the only solution for α_1, α_2 and α_3 is the zero solution. The left-hand side of the equation equals

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Hence we solve

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which forces

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent.

Example 3.9 *Show that*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a linearly independent set of vectors in the vector space $M_{2\times 2}(F)$ *of* 2×2 *matrices over the field* F*.*

Solution: We solve

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

that is,

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\alpha = \beta = 0$ and we conclude that our set is linearly independent.

Example 3.10 *Show that the following set of vectors in* $\mathbb{R}^3$ *is linearly dependent:*

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Solution: Note that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Hence one vector in $\mathscr{A}$ can be expressed as a linear combination of the others, so $\mathscr{A}$ is linearly dependent.

If we rearrange this equation we arrive at

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}.$$

This equation also shows that $\mathscr{A}$ is not linearly independent since we have non-zero solutions to the equation expressing linear dependence.

Example 3.11 *Determine whether the set*

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a linearly independent set in $\mathbb{R}^3$*.*

Solution: We attempt to solve

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_3 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0};$$

that is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Clearly $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is a solution to this equation. We are interested in whether this is the only solution. One way of proceeding would be to simply apply the usual method (from MT1002) for solving such systems of linear equations. Indeed, this will produce a perfectly valid solution.

An alternative method, however, is to make use of Theorem 1.19 to tell us that Equation (3.1) has a solution if and only if the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

is invertible. We calculate

$$\begin{aligned} \det M &= \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} - \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= 1 - 2 + (4 - 1) \\ &= 2 \neq 0. \end{aligned}$$

Hence M is invertible, so Equation (3.1) has a unique solution, namely $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore $\mathscr{A}$ is linearly independent.

Bases

We now put together the two concepts introduced so far in this chapter.

Definition 3.12 Let V be a vector space over a field F. A subset $\mathscr{B}$ of V is called a *basis* if

(i) $\mathscr{B}$ is a spanning set for V, and

(ii) $\mathscr{B}$ is linearly independent.

Throughout this course, we shall assume that any vector space we work with has a finite basis (that is, a basis containing only a finite number of vectors). Linear algebra can be done with infinite bases, but we shall avoid such complications here.

Theorem 3.13 *Let $\mathscr{B}$ be a basis for a vector space V. Then every vector in V can be expressed in* precisely one way *as a linear combination of the vectors in the basis $\mathscr{B}$.*

PROOF: Suppose that $\mathscr{B} = \{v_1, v_2, \dots, v_k\}$. Since $\mathscr{B}$ is, in particular, a spanning set for V, every vector in V is a linear combination of the vectors in $\mathscr{B}$. We need to use the fact that $\mathscr{B}$ is linearly independent to show that every such linear combination is unique.

Let $v \in V$ and suppose that we have two linear combination expressions for v in terms of $\mathscr{B}$, say

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k. \tag{3.2}$$

Rearranging the terms we obtain

$$(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_k - \beta_k)v_k = \mathbf{0}.$$

However, $\mathscr{B}$ is linearly independent, so the only possible way that a linear combination of the vectors in $\mathscr{B}$ can equal the zero vector $\mathbf{0}$ is if all coefficients involved are zero. Hence

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_k - \beta_k = 0;$$

that is,

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2, \quad \ldots, \quad \alpha_k = \beta_k.$$

Hence the coefficients occurring in Equation (3.2) are the same and we conclude that every linear combination expression for v in terms of the basis $\mathscr{B}$ is indeed unique. □

Example 3.14 The set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form a basis for $\mathbb{R}^3$. Indeed, we saw in Example 3.3 that this set spans $\mathbb{R}^3$, while Example 3.8 tells us that it is linearly independent.

We also call this set the *standard basis* for $\mathbb{R}^3$.

Example 3.15 *Show that* $\{1, x, x^2\}$ *forms a basis for the space* $\mathcal{P}_2$ *of all polynomials of degree at most* 2 *with real coefficients.*

Solution: Let $\mathscr{B} = \{1, x, x^2\}$. First note that if $f(x)$ is any polynomial in $\mathcal{P}_2$, then it has the form

$$f(x) = a_0 + a_1 x + a_2 x^2$$

and so $f(x)$ is a linear combination of the polynomials 1, x and x^2. Hence $\mathscr{B}$ is a spanning set for $\mathcal{P}_2$.

To determine linear independence, note that $a_0 + a_1 x + a_2 x^2 = 0$ if and only if $a_0 = a_1 = a_2 = 0$. Hence $\mathscr{B}$ is a linearly independent set.

Hence $\mathscr{B}$ is a basis for $\mathcal{P}_2$.

Example 3.16 *Show that*

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for the vector space $M_{2\times 2}(F)$ *of all* 2×2 *matrices over the field* F.

Solution: First note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so every matrix in $M_{2\times 2}(F)$ is a linear combination of the set $\mathscr{B}$. Hence $\mathscr{B}$ is a spanning set for our vector space.

Secondly if

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

that is, if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then $a = b = c = d = 0$. This demonstrates that $\mathscr{B}$ is a linearly independent set.

Hence $\mathscr{B}$ is a basis for $M_{2\times 2}(F)$.

The next example shows that a basis for a vector space V need not be unique. (Indeed, very few vector spaces have a unique basis.)

Example 3.17 *Consider the following three polynomials*

$$p_1(x) = 1 + 2x + x^2, \qquad p_2(x) = 1 + x + 2x^2, \qquad p_3(x) = 1 + x^2.$$

Show that $\mathscr{C} = \{p_1(x), p_2(x), p_3(x)\}$ *is a basis for* $\mathcal{P}_2$.

Comparing with Example 3.15 shows that the natural basis for a space is not necessarily the only basis.

Solution: We seek to show that every polynomial can be expressed in the form

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x)$$

(for this shows that $\mathscr{C}$ spans $\mathcal{P}_2$) and, moreover, that the coefficients α, β and γ are uniquely determined. The latter will ensure linear independence, since when we solve

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = 0 = 0 \cdot p_1(x) + 0 \cdot p_2(x) + 0 \cdot p_3(x)$$

the uniqueness forces $\alpha = \beta = \gamma = 0$, as required.

So let $f(x) = a + bx + cx^2 \in \mathcal{P}_2$ and solve

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = f(x); \tag{3.3}$$

that is,

$$\alpha(1 + 2x + x^2) + \beta(1 + x + 2x^2) + \gamma(1 + x^2) = a + bx + cx^2.$$

Equating coefficients gives the system of three linear equations:

$$\begin{aligned} \alpha + \beta + \gamma &= a \\ 2\alpha + \beta &= b \\ \alpha + 2\beta + \gamma &= c; \end{aligned}$$

that is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We know that this has a unique solution if and only if the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

is invertible (i.e., has non-zero determinant), by Theorem 1.19. Indeed,

$$\det A = 1 - 2 + (4 - 1) = 2 \neq 0,$$

so there is indeed a unique solution to the Equation (3.3).

We conclude that $\mathscr{C}$ is indeed a linearly independent spanning set, that is, a basis for $\mathcal{P}_2$.

The basic fact that one needs to know about bases for vector spaces is that they always exist. Indeed, if one starts with a spanning set for a vector space, then one can produce a basis by omitting the correct choice of vectors.

Theorem 3.18 *Let V be a vector space and let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in V.*

(i) *If $\mathscr{A}$ spans V, then there is some subset of $\mathscr{A}$ which is a basis for V.*

(ii) *If $\mathscr{A}$ is linearly independent, then we can adjoin more vectors to $\mathscr{A}$ to produce a set $\mathscr{B}$ which is a basis for V and contains $\mathscr{A}$.*

We omit the proof. The essential idea for the first is that if $\mathscr{A}$ is not linearly independent, then some vector v_i in $\mathscr{A}$ is a linear combination of the other vectors. If we omit v_i then we show that the resulting set also spans V. Repeating this process produces a set which is linearly independent and spans V; that is, a basis for V.

Similarly, for the second the idea is that if $\mathscr{A}$ does not span V, then there is some vector which is not a linear combination of the vectors in $\mathscr{A}$. Adjoining this vector will still give a linear independent set. Repeating this process produces a basis when V is finite-dimensional. We summarise the procedure by saying that every linearly independent set can be *extended* to a basis for the vector space.

(More details will be provided in MT3501 for both facts.)

Dimension

Definition 3.19 Let V be a vector space over a field F. We say that V is *finite-dimensional* if it possesses a finite spanning set; that is, if V possesses a finite basis. The *dimension* of V is the size of any basis for V and is denoted by $\dim V$.

For this definition to make sense, we need to know that the dimension is uniquely determined by the vector space; i.e., that it is not possible for a vector space to have bases of different sizes. We state this as a theorem, though we defer the proof to MT3501.

Theorem 3.20 *Every basis for a finite-dimensional vector space contains the same number of vectors.*

Example 3.21 The dimension of $\mathbb{R}^3$ as a vector space over $\mathbb{R}$ is 3. Indeed, in Example 3.14 we observed that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$.

Example 3.22 Example 3.15 tells us that $\mathcal{P}_2$ is a vector space of dimension 3 over $\mathbb{R}$.

Example 3.23 Example 3.16 tells us that $M_{2\times 2}(F)$ is a vector space of dimension 4 over the field F.

Example 3.24 Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = 0.$$

From MT1002, you will know know that the general equation to this equation is

$$y = A\mathrm{e}^x + B\mathrm{e}^{-x}$$

where A and B are any constants. This tells us that the set of all functions which are solutions to the differential equation forms a vector space over $\mathbb{R}$ and, moreover, that every solution can be written uniquely as a linear combination of the two functions $f_1(x) = \mathrm{e}^x$ and $f_2(x) = \mathrm{e}^{-x}$. Hence these two functions form a basis for the vector space of solutions and we conclude that the solution space has dimension 2.

This is a general phenomenon of solutions to linear differential equations. A linear differential equation of degree n will typically have a solution space of dimension n.

Chapter 4

Linear Transformations

Linear transformations (also frequently called linear mappings) are functions between vector spaces that have particularly nice properties. They actually occur very widely in mathematics and its applications. In this chapter we shall investigate the general properties of linear transformations, in particular noting the close connection between them and matrices.

Definition 4.1 Let V and W be two vector space over the same field F of scalars. Consider a mapping $T\colon V \to W$; that is, to each vector $v \in V$ the mapping associates a vector $T(v)$ in W. We shall call T a *linear transformation* or *linear mapping* if the following two conditions hold:

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$, and

(ii) $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and all scalars $\alpha \in F$.

Example 4.2 One of the most standard examples of a linear transformation is to take an $m \times n$ matrix A, say with real entries, and to define a mapping $\mathbb{R}^n \to \mathbb{R}^m$ by multiplying by A:

$$\boldsymbol{v} \mapsto A\boldsymbol{v}.$$

This is a linear transformation due to standard properties of matrix multiplication:

$$A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$$

and

$$A(\alpha\boldsymbol{v}) = \alpha \cdot A\boldsymbol{v}$$

(for any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and any $\alpha \in \mathbb{R}$) are well-known (and easily verified) properties.

Hence multiplication by A is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

Example 4.3 *Define a function* $T\colon M_{2\times 2}(F) \to M_{2\times 1}(F)$ *by*

$$T\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix}.$$

Show that T *is linear.*

Solution: We check the conditions:

$$\begin{aligned} T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) &= T\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1+a_2 \\ c_1+c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = T\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + T\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{aligned}$$

and

$$T\left(\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = T\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha c \end{pmatrix} = \alpha\begin{pmatrix} a \\ c \end{pmatrix} = \alpha \cdot T\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence T is a linear transformation.

Example 4.4 *Recall that* $\mathcal{P}_n$ *denotes the space of real polynomials of degree at most* n. *Define* $T\colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ *by*

$$T\colon p \mapsto \frac{\mathrm{d}p}{\mathrm{d}x}.$$

Show that T *is a linear transformation.*

Solution: We simply use two standard properties of differentiation. If p and q are polynomials (indeed, any differentiable functions) we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}(p+q) = \frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\mathrm{d}q}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\alpha p) = \alpha\frac{\mathrm{d}p}{\mathrm{d}x};$$

that is,

$$T(p+q) = T(p) + T(q) \qquad \text{and} \qquad T(\alpha p) = \alpha T(p)$$

for any $p, q \in \mathcal{P}_n$ and any $\alpha \in \mathbb{R}$, as required.

Example 4.5 *Define* $T\colon \mathbb{R} \to \mathbb{R}^3$ *by*

$$T(x) = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}.$$

Show that T *is a linear transformation.*

Solution:

$$T(x+y) = \begin{pmatrix} x+y \\ 2(x+y) \\ 3(x+y) \end{pmatrix} = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} + \begin{pmatrix} y \\ 2y \\ 3y \end{pmatrix} = T(x) + T(y)$$

and

$$T(\alpha x) = \begin{pmatrix} \alpha x \\ 2\alpha x \\ 3\alpha x \end{pmatrix} = \alpha \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} = \alpha T(x).$$

Hence T is linear.

The following describes the basic properties of a linear transformation. They are all direct consequences of the definition.

Proposition 4.6 *Let $T\colon V \to W$ be a linear transformation between two vector spaces over the field F. Then*

(i) $T(\mathbf{0}) = \mathbf{0}$*; that is, the zero vector in V is mapped to the zero vector in W by T;*

(ii) $T(-v) = -T(v)$ *for all $v \in V$;*

(iii) *if $v_1, v_2, \dots, v_k \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars in F, then*

$$T(\alpha_1 v_1 + \dots + \alpha_k v_k) = \alpha_1 T(v_1) + \dots + \alpha_k T(v_k).$$

PROOF: (i) We know that $0 \cdot \mathbf{0} = \mathbf{0}$ (by Proposition 2.6), so applying T gives

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$$

(using Proposition 2.6(ii) again).

(ii) Proposition 2.6(iv) tells us that $(-1)v = -1v = -v$, so

$$T(-v) = T((-1)v) = (-1)T(v) = -T(v).$$

(iii) Expanding using repeated application of the two conditions of Definition 4.1, we see

$$T(\alpha_1 v_1 + \dots + \alpha_k v_k) = T(\alpha_1 v_1) + \dots + T(\alpha_k v_k) = \alpha_1 T(v_1) + \dots + \alpha_k T(v_k).$$

□

The matrix of a linear transformation

Proposition 4.7 *Let V and W be vector spaces over a field F. Then any linear transformation $T\colon V \to W$ is uniquely determined by its effect on a basis for V.*

PROOF: Suppose that $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V. We shall show that if $w_i = T(v_i)$ is known for each i, then the effect of T on any vector in V is uniquely determined.

If $v \in V$, then as $\mathscr{B}$ is a basis, there exist unique scalars α_i such that

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

(see Theorem 3.13). Consequently, when we use Proposition 4.6(iii), we see

$$\begin{aligned} T(v) &= T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\ &= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) \\ &= \alpha_1 w_1 + \cdots + \alpha_n w_n. \end{aligned}$$

Hence the effect of T on every vector in V is uniquely specified. □

Example 4.8 *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that*

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \textit{and} \qquad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

What is the value

$$T\begin{pmatrix}a\\b\end{pmatrix}?$$

Solution: Note that

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix},$$

so by linearity

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

We are now able to calculate the effect of T on an arbitrary vector in $\mathbb{R}^2$:

$$\begin{aligned} T\begin{pmatrix}a\\b\end{pmatrix} &= T\left(a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= aT\begin{pmatrix}1\\0\end{pmatrix} + bT\begin{pmatrix}0\\1\end{pmatrix} \end{aligned}$$

$$= a\begin{pmatrix}-1\\1\\-1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}b-a\\a\\b-a\end{pmatrix}.$$

Example 4.9 We shall now give an example to illustrate how the original linear transformation can be recovered from the action on a basis.

Define a linear transformation $T\colon \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ by

$$T\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a & a+b\\ c & a+c\end{pmatrix}.$$

The effect of T on the standard basis for $\mathbb{R}^3$ is then

$$T(\mathbf{e}_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$T(\mathbf{e}_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$T(\mathbf{e}_3) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}.$$

Now let us exploit linearity to reconstruct the general rule:

$$\begin{aligned}T\begin{pmatrix}a\\b\\c\end{pmatrix} &= T(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3)\\ &= aT(\mathbf{e}_1) + bT(\mathbf{e}_2) + cT(\mathbf{e}_3)\\ &= a\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\\ &= \begin{pmatrix}a & a+b\\ c & a+c\end{pmatrix}.\end{aligned}$$

We shall now describe how a matrix can be associated to any linear transformation from one vector to another.

Let V and W be finite-dimensional vector spaces over the same field F. Suppose that $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_m\}$ are bases for V and W, respectively, so that $\dim V = n$ and $\dim W = m$. (Strictly speaking we are working with *ordered bases* here, since the indexing of the basis vectors matters and reordering the vectors will change the matrix we produce.)

Now let $T\colon V \to W$ be a linear transformation. We know that T is uniquely determined by the images $T(v_1)$, $T(v_2)$, ..., $T(v_n)$ of the basis vectors. These images are vectors in W, so we can uniquely express each one in terms of the basis $\mathscr{C}$ for W:

$$\begin{aligned} T(v_1) &= \alpha_{11}w_1 + \alpha_{21}w_2 + \cdots + \alpha_{m1}w_m \\ T(v_2) &= \alpha_{12}w_1 + \alpha_{22}w_2 + \cdots + \alpha_{m2}w_m \\ \vdots \quad & \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ T(v_n) &= \alpha_{1n}w_1 + \alpha_{2n}w_2 + \cdots + \alpha_{mn}w_m \end{aligned}$$

That is, the general formula is

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij}w_i \qquad \text{for } j = 1,\ 2,\ \ldots,\ n.$$

(Take note of the order of subscripts appearing on the coefficients here. They are not quite in the order one might expect!)

Definition 4.10 The $m \times n$ matrix $[\alpha_{ij}]$ whose (i,j)th entry is the coefficient α_{ij} appearing above is called the *matrix of T with respect to the bases $\mathscr{B}$ and $\mathscr{C}$*.

There is no standard notation for this matrix, but we shall choose to denote it by $\mathrm{Mat}(T)$ or, when we wish to be explicit about the dependence on the bases $\mathscr{B}$ and $\mathscr{C}$, by $\mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T)$.

Note that this matrix very much depends on the choice of bases for the two vector spaces concerned. The crucial point to remember when constructing this matrix is that the first *column* consists of those coefficients that arise when we write $T(v_1)$ in terms of the basis $\mathscr{C}$. More generally, the scalars appearing in the jth *column* are the coefficients that arise when we write $T(v_j)$ in terms of the basis $\mathscr{C}$.

Example 4.11 *Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ given by*

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+2y+z \\ y+3z \end{pmatrix}.$$

What is the matrix of T with respect to the standard bases for the vector spaces?

Solution: We calculate the effect of T on each of the basis vectors for $\mathbb{R}^3$ and find the coefficients when these images are written in terms of the basis for $\mathbb{R}^4$:

$$T(\mathbf{e}_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$T(\mathbf{e}_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\2\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$T(\mathbf{e}_3) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\3\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We now write the coefficients appearing down the column of the matrix. Thus the matrix of T with respect to the standard bases is

$$\operatorname{Mat}(T) = \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\\0&1&3\end{pmatrix}.$$

So what does the matrix of a linear transformation do? If $T\colon V \to W$ is a linear transformation, $n = \dim V$, $m = \dim W$ and $A = \operatorname{Mat}(T)$, then A gives us a linear transformation $A\colon F^n \to F^m$ (by multiplying vectors by the matrix). Moreover, A has the same effect on the standard bases for F^n and F^m as T does on the bases for V and W that we are considering. In particular, when we calculate the matrix A of a linear transformation $T\colon F^n \to F^m$ with respect to the standard bases, then T actually is the same as multiplication by the matrix A.

We can see this in the context of the above example. Indeed, for the matrix

$$A = \operatorname{Mat}(T) = \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\\0&1&3\end{pmatrix},$$

we calculate

$$A\boldsymbol{v} = A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\\0&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\x+y\\x+2y+z\\y+3z\end{pmatrix} = T(\boldsymbol{v}).$$

Example 4.12 *Recall that $\mathcal{P}_n$ denotes the space of polynomials of degree at most n with real coefficients. Let $T\colon \mathcal{P}_n \to \mathcal{P}_{n-2}$ be the linear transformation given by*

$$T\colon p \mapsto \frac{\mathrm{d}^2 p}{\mathrm{d}x^2} \qquad \text{for a polynomial } p.$$

Find the matrix of T relative to some bases for $\mathcal{P}_n$ and $\mathcal{P}_{n-2}$.

Solution: A natural basis for $\mathcal{P}_n$ is $\{1, x, x^2, x^3, \dots, x^n\}$ consisting of all *monomials.* A similar basis exists for $\mathcal{P}_{n-2}$. We calculate the effect of T on each element of the basis for $\mathcal{P}_n$ and write it in terms of the basis for $\mathcal{P}_{n-2}$:

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 0 \\ T(x^2) &= 2 \\ T(x^3) &= 0 + 6x \\ T(x^4) &= 0 + 0x + 12x^2 \\ &\vdots \\ T(x^n) &= 0 + 0x + \dots + 0x^{n-1} + n(n-1)x^{n-2} \end{aligned}$$

We now write the coefficients appearing down the columns of our matrix:

$$\mathrm{Mat}(T) = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 6 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 12 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & n(n-1) \end{pmatrix}$$

Example 4.13 (September 2003) *A mapping T from $\mathbb{R}^2$ to itself is defined by*

$$T\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 2y \\ x - y \end{pmatrix}.$$

Show that this is a linear transformation. Find the matrix of this transformation with respect to the standard basis for $\mathbb{R}^2$.

Solution: We check the two conditions for linearity:

$$\begin{aligned} T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} &= \begin{pmatrix} x_1 + x_2 + 2(y_1 + y_2) \\ x_1 + x_2 - (y_1 + y_2) \end{pmatrix} \\ &= \begin{pmatrix} x_1 + 2y_1 \\ x_1 - y_1 \end{pmatrix} + \begin{pmatrix} x_2 + 2y_2 \\ x_2 - y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \end{aligned}$$

and

$$T\left(\alpha\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}\alpha x\\\alpha y\end{pmatrix} = \begin{pmatrix}\alpha x + 2\alpha y\\\alpha x - \alpha y\end{pmatrix} = \alpha\begin{pmatrix}x+2y\\x-y\end{pmatrix} = \alpha T\begin{pmatrix}x\\y\end{pmatrix}.$$

Hence T is a linear transformation.

Now we apply T to the standard basis of $\mathbb{R}^2$:

$$T(\mathbf{e}_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \mathbf{e}_1 + \mathbf{e}_2$$
$$T(\mathbf{e}_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} = 2\mathbf{e}_1 - \mathbf{e}_2$$

Hence the matrix of T with respect to the standard basis is

$$\mathrm{Mat}(T) = \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}.$$

Rank and nullity

Definition 4.14 Let $T\colon V \to W$ be a linear transformation.

(i) The *image* of T consists of all images $T(v)$ of vectors in V under T. It is denoted by $\operatorname{im} T$ or by $T(V)$:

$$\operatorname{im} T = T(V) = \{\, T(v) \mid v \in T \,\}.$$

(ii) The *kernel* or *nullspace* of T consists of all vectors in V which are mapped to the zero vector of W by T. It is denoted by $\ker T$:

$$\ker T = \{\, v \in V \mid T(v) = \mathbf{0} \,\}.$$

Proposition 4.15 *Let $T\colon V \to W$ be a linear transformation, where V and W are vector spaces over a field F.*

(i) *If U is a subspace of V, then $T(U) = \{\, T(u) \mid u \in U \,\}$ is a subspace of W.*

(ii) *The image of T is a subspace of W.*

(iii) *The kernel of T is a subspace of V.*

PROOF: (i) Certainly $T(U)$ is non-empty since U is non-empty. Now let $w_1, w_2 \in T(U)$. Then $w_1 = T(u_1)$ and $w_2 = T(u_2)$ for some $u_1, u_2 \in U$. Hence

$$w_1 + w_2 = T(u_1) + T(u_2) = T(u_1 + u_2) \in T(U),$$

since U is a subspace of V so $u_1 + u_2 \in U$. Similarly if $\alpha \in F$ and $w \in T(U)$, then $w = T(u)$ for some $u \in U$. We calculate that

$$\alpha w = \alpha T(u) = T(\alpha u) \in T(U),$$

since U is a subspace of V so $\alpha u \in U$. We have shown than $T(U)$ is closed under addition and scalar multiplication. We conclude that $T(U)$ is a subspace of W.

(ii) This follows immediately from (i) since it is the special case that $U = V$.

(iii) We know that $T(\mathbf{0}) = \mathbf{0}$ and therefore $\mathbf{0} \in \ker T$. We are at least therefore dealing with a non-empty subset of V. Now let $v_1, v_2 \in \ker T$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and we deduce that $v_1 + v_2 \in \ker T$. Now let $\alpha \in F$ and $v \in \ker T$. Then

$$T(\alpha v) = \alpha T(v) = \alpha \mathbf{0} = \mathbf{0}$$

and we deduce that $\alpha v \in \ker T$. Hence $\ker T$ is closed under addition and scalar multiplication, so it is a subspace of V. □

Now that we know that the image and the kernel of $T \colon V \to W$ are subspaces of W and V, respectively, it makes sense to talk about their dimensions. Consequently, we can make the following definition:

Definition 4.16 Let $T \colon V \to W$ be a linear transformation.

(i) The *rank* of T is the dimension of the image $\operatorname{im} T$ of T. We shall denote this by $\operatorname{rank} T$.

(ii) The *nullity* of T is the dimension of the kernel $\ker T$ of T. We shall denote this by $\operatorname{null} T$.

Comment: The notations here are not uniformly established and they are selected for convenience rather than for being definitive. Many authors use different notations or, more commonly, no specific notation whatsoever for these two concepts.

Example 4.17 *Define the linear map $T \colon \mathbb{R} \to \mathbb{R}^3$ by*

$$T(x) = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}.$$

Find the rank and nullity of T.

Solution: The image of T is

$$\begin{aligned}
\operatorname{im} T &= \{\, T(x) \mid x \in \mathbb{R} \,\} \\
&= \left\{ \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \\
&= \left\{ x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\},
\end{aligned}$$

the set of all scalar multiples of the vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Hence $\operatorname{im} T$ is the subspace spanned by the set consisting of this single non-zero vector. This set is linearly independent (a set consisting of a single non-zero vector is always linearly independent) and so we conclude that it is a basis for $\operatorname{im} T$. Hence the rank of T is 1. (The image of T has dimension 1.)

The kernel of T consists of those $x \in \mathbb{R}$ such that $T(x) = \mathbf{0}$. Now

$$T(x) = \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Longleftrightarrow \qquad x = 0.$$

Hence $\ker T = \{0\}$, which is a zero dimensional space. Hence the nullity of T is 0.

Example 4.18 *Find the rank and nullity of the linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *given by*

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x + y \\ x - y \end{pmatrix}.$$

Solution: A vector in the image of T has the form

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x + y \\ x - y \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence the set

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a spanning set for $\operatorname{im} T$. However, this set is also linearly independent, for if

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$\begin{pmatrix} \alpha \\ \alpha + \beta \\ \alpha - \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and we deduce $\alpha = \beta = 0$. Hence $\mathscr{A}$ is a basis for $\operatorname{im} T$ and so

$$\operatorname{rank} T = \dim \operatorname{im} T = 2.$$

The kernel of T consists of those vectors $\boldsymbol{v}$ for which $T(\boldsymbol{v}) = \mathbf{0}$; that is, we solve

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x + y \\ x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and we deduce that $x = y = 0$. Hence

$$\ker T = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \{\mathbf{0}\},$$

which is a zero-dimensional space. Therefore $\operatorname{null} T = 0$.

Notice that in Example 4.17 that $T \colon \mathbb{R} \to \mathbb{R}^3$ satisfies

$$\operatorname{rank} T + \operatorname{null} T = 1 + 0 = 1 = \dim \mathbb{R}$$

while in Example 4.18 our transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$\operatorname{rank} T + \operatorname{null} T = 2 + 0 = 2 = \dim \mathbb{R}^2.$$

These are actually examples of a general phenomenon.

Theorem 4.19 (Rank-Nullity Theorem) *Let $T \colon V \to W$ be a linear transformation between finite-dimensional vector spaces over a field F. Then*

$$\operatorname{rank} T + \operatorname{null} T = \dim V.$$

PROOF: Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for $\ker T$ (so that $n = \operatorname{null} T$) and extend this to a basis $\mathscr{C} = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$ for V (so that $\dim V = n + k$). We now seek to find a basis for $\operatorname{im} T$.

If $w \in \operatorname{im} T$, then $w = T(v)$ for some $v \in V$. We can write v as a linear combination of the vectors in the basis $\mathscr{C}$, say

$$v = \alpha_1 v_1 + \dots + \alpha_{n+k} v_{n+k}$$

for some scalars $\alpha_1, \dots, \alpha_{n+k} \in F$. Then, applying T and using linearity,

$$\begin{aligned} w = T(v) &= T(\alpha_1 v_1 + \dots + \alpha_{n+k} v_{n+k}) \\ &= \alpha_1 T(v_1) + \dots + \alpha_{n+k} T(v_{n+k}) \\ &= \alpha_{n+1} T(v_{n+1}) + \dots + \alpha_{n+k} T(v_{n+k}), \end{aligned}$$

since $T(v_1) = \dots = T(v_n) = \mathbf{0}$ as $v_1, \dots, v_n \in \ker T$. This shows that the set $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ spans $\operatorname{im} T$.

Now we show that $\mathscr{D}$ is linearly independent. Suppose that

$$\beta_1 T(v_{n+1}) + \dots + \beta_k T(v_{n+k}) = \mathbf{0};$$

that is

$$T(\beta_1 v_{n+1} + \dots + \beta_k v_{n+k}) = \mathbf{0}.$$

Hence the vector

$$\beta_1 v_{n+1} + \dots + \beta_k v_{n+k}$$

belongs to the kernel of T. We know that $\mathscr{B}$ is a basis for $\ker T$ and hence

$$\beta_1 v_{n+1} + \dots + \beta_k v_{n+k} = \gamma_1 v_1 + \dots + \gamma_n v_n$$

for some $\gamma_1, \dots, \gamma_n \in F$. Rearranging we obtain the equation

$$(-\gamma_1) v_1 + \dots + (-\gamma_n) v_n + \beta_1 v_{n+1} + \dots + \beta_k v_{n+k} = \mathbf{0}.$$

This equation involves the vectors in the basis $\mathscr{C}$ for V. Therefore, since $\mathscr{C}$ is linearly independent, we conclude that all the coefficients involved are zero. In particular,

$$\beta_1 = \beta_2 = \dots = \beta_k = 0,$$

which is what we needed to deduce that $\mathscr{D}$ is linearly independent.

Hence $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ is a basis for $\operatorname{im} T$ and so

$$\operatorname{rank} T = \dim \operatorname{im} T = k = (n+k) - n = \dim V - \operatorname{null} T.$$

Thus

$$\operatorname{rank} T + \operatorname{null} T = \dim V,$$

as claimed. □

The advantage of this theorem is that it tells us that once we know either the rank or the nullity of a linear transformation, then we can (pretty much immediately) deduce the other. This, of course, saves us a considerable amount of work.

The rank of a matrix

We can also simplify the process of finding the rank of a linear transformation by exploiting its matrix with respect to some bases. This involves what is known as the rank of a matrix, which we shall now describe.

Definition 4.20 Let A be an $m \times n$ matrix.

(i) The columns of A can be viewed as (column) vectors in F^m. The subspace of F^m spanned by these columns is called the *column-space* of A.

The *column-rank* of A is the dimension of its column-space.

(ii) The rows of A can be viewed as (row) vectors in F^n. The subspace of F^n spanned by these rows is called the *row-space* of A.

The *row-rank* of A is the dimension of its row-space.

Example 4.21 Consider the 3×3 identity matrix I. Its column-space is spanned by its three columns

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(viewed as elements of $\mathbb{R}^3$). We have already observed that these three vectors form a basis for $\mathbb{R}^3$. Hence the column-space is the whole of $\mathbb{R}^3$ and, since $\dim \mathbb{R}^3 = 3$, the column-rank of I is 3.

Theorem 4.22 *Let A be an $m \times n$ matrix over a field F. The following numbers are equal:*

(i) *the row-rank of A;*

(ii) *the column-rank of A;*

(iii) *the rank of the linear transformation $A \colon F^n \to F^m$ given by multiplication by A; that is, $\boldsymbol{v} \mapsto A\boldsymbol{v}$.*

PROOF: We omit the proof that (i) and (ii) are equal. Full details can be found, for example, on page 47 of Blyth & Robertson.

To show that (ii) and (iii) are equal, we shall simply show that the column-space and the image of the linear transformation $\boldsymbol{v} \mapsto A\boldsymbol{v}$ are the *same subspace* of F^m. Hence their dimensions are the same, which is what we need to show. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ denote the columns of A, so A has the form

$$A = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix}.$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$ be the standard basis for F^n; that is,

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with the 1 in the ith entry. Thus an arbitrary vector in F^n has the form

$$\boldsymbol{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n.$$

Hence

$$\begin{aligned} A\boldsymbol{v} &= A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n) \\ &= x_1\, A\mathbf{e}_1 + x_2\, A\mathbf{e}_2 + \dots + x_n\, A\mathbf{e}_n \end{aligned}$$

and

$$A\mathbf{e}_i = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} \mathbf{e}_i = \boldsymbol{a}_i.$$

So

$$A\boldsymbol{v} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \dots + x_n\boldsymbol{a}_n.$$

Therefore

$$\begin{aligned} \operatorname{im} A &= \{\, A\boldsymbol{v} \mid \boldsymbol{v} \in F^n \,\} \\ &= \{\, x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \dots + x_n\boldsymbol{a}_n \mid x_1, x_2, \dots, x_n \in F \,\} \\ &= \operatorname{Span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n). \end{aligned}$$

Hence the image of A and the column-space are equal, so the rank and column-rank are equal. □

In view of this result, we tend to simply refer to the *rank* of a matrix. To find the rank of a matrix, we make use of the familiar row operations. By applying row operations, we can reduce any matrix to what is known as echelon form.

Definition 4.23 A matrix E is said to be in *echelon form* if

(i) all non-zero rows of E are above any rows consisting entirely of zeros and

(ii) each leading (non-zero) entry in a row is in a column to the right of the leading entry of the row above it.

Thus a matrix in echelon form has the following general shape:

$$E = \begin{pmatrix} 0 & 0 & * & * & \cdots & & \\ 0 & \cdots & \cdots & 0 & * & \cdots & \\ 0 & \cdots & \cdots & \cdots & 0 & * & \cdots \\ \vdots & & & & \ddots & 0 & * \\ 0 & 0 & \cdots & \cdots & & & 0 \end{pmatrix}$$

Theorem 4.24 *If a matrix A is row-equivalent to a matrix E in echelon form, then the rank of A is equal to the number of non-zero rows in E.*

Example 4.25 *Find the rank of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 7 \end{pmatrix}.$$

Solution: We first apply row operations to reduce A into echelon form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -2 \end{pmatrix} \qquad r_2 \mapsto r_2 - 2r_1,\ r_3 \mapsto r_3 - 3r_1$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad r_3 \mapsto r_3 + 2r_2.$$

This matrix in echelon form has three non-zero rows. We conclude that A has rank 3.

Rank and the matrix of a linear transformation

Given a linear transformation $T \colon V \to W$ of two finite-dimensional vector spaces V and W, we have already described how to construct the matrix of the linear transformation. Suppose that $\dim V = n$ and $\dim W = m$. Let $A = \mathrm{Mat}(T)$ be the matrix of T with respect to some bases for V and W. Then A is an $m \times n$ matrix which encodes the linear transformation T. In particular, the way that T behaves is essentially the same as the way that $\boldsymbol{v} \mapsto A\boldsymbol{a}$ behaves as a linear transformation $A \colon F^n \to F^m$. In particular, the following fact is true:

Theorem 4.26 *Let $T \colon V \to W$ be a linear transformation and $A = \mathrm{Mat}(T)$ with respect to some bases for V and W. Then*

$$\operatorname{rank} T = \operatorname{rank} A.$$

Theorem 4.22 tells us that the rank of A as a linear transformation is equal to both its row-rank and its column-rank. We therefore have the following recipe for calculating the rank and nullity of a linear transformation.

- Let $T \colon V \to W$ be a linear transformation.
- Determine $A = \mathrm{Mat}(T)$, with respect to some bases.
- Apply row operations to produce a matrix E in echelon form that is row-equivalent to A.
- The rank of T is equal to the rank of A and this equals the number of non-zero rows in E.
- Finally determine the nullity of T via the Rank-Nullity Theorem (that is, Theorem 4.19).

This is usually a reasonably straightforward algorithmic process. Nevertheless it is often not that much easier than determining a basis explicitly for either the kernel or the image of T.

Example 4.27 *Find the rank, nullity, kernel and a basis for the kernel of the linear transformation* $\mathbb{R}^4 \to \mathbb{R}^3$ *given by* $\boldsymbol{v} \mapsto A\boldsymbol{v}$ *where*

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{pmatrix}.$$

Solution: The matrix of this linear transformation with respect to the standard basis of the vector spaces is the original matrix A. We apply row operations to reduce it to echelon form:

$$\begin{aligned} A &\to \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{pmatrix} & \qquad & \begin{array}{l} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 + r_1 \end{array} \\ &\to \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} & & r_3 \mapsto r_3 - r_2 \end{aligned}$$

This last matrix is in echelon form and has two non-zero rows. We conclude that our original matrix has rank 2. Then by the Rank-Nullity Theorem, $\operatorname{rank} A + \operatorname{null} A = \dim \mathbb{R}^4 = 4$. Hence the nullity of A equals 2.

To find the kernel, we solve $A\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is a system of linear equations which we solve by applying Gaussian elimination to:

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 2 & 4 & 3 & 5 & 0 \\ -1 & -2 & 6 & 7 & 0 \end{array}\right)$$

But Gaussian elimination is performed by precisely the row operations we used above, so we reduce to

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 0 & 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Hence we solve

$$x + 2y - z + 4t = 0 \quad \text{and} \quad 5z - 3t = 0.$$

The solutions to these equations is parametrized by y and t and are given by $z = \frac{3}{5}t$ and $x = -2y - \frac{17}{5}t$. Therefore

$$\ker A = \left\{ \begin{pmatrix} -2y - \frac{17}{5}t \\ y \\ \frac{3}{5}t \\ t \end{pmatrix} \;\middle|\; y, t \in \mathbb{R} \right\}.$$

An arbitrary vector in the kernel has the form

$$\begin{pmatrix} -2y - \frac{17}{5}t \\ y \\ \frac{3}{5}t \\ t \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -17/5 \\ 0 \\ 3/5 \\ 1 \end{pmatrix}.$$

We conclude that every vector in the kernel is a linear combination of the two vectors appearing on the right-hand side here and it is easy to check they are linearly independent. Hence

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -17/5 \\ 0 \\ 3/5 \\ 1 \end{pmatrix} \right\}$$

is a basis for the kernel of A.

Chapter 5

Eigenvalues, Eigenvectors and Diagonalisation

In this section we shall be principally concerned with the situation that we have a vector space V and a linear transformation $T\colon V \to V$ (that is, from V back to itself). A particular case will be when we have a square $n \times n$ matrix A viewed as a linear transformation $A\colon F^n \to F^n$.

Eigenvalues and eigenvectors

Definition 5.1 Let V be a vector space and $T\colon V \to V$ be a linear transformation. We say that a scalar λ is an *eigenvalue* of T if there is some *non-zero* vector v in V such that

$$Tv = \lambda v.$$

Any non-zero vector satisfying this equation will be called an *eigenvector* for T associated to the eigenvalue λ.

In particular, if A is a square $n \times n$ matrix with entries from a field F, then $\lambda \in F$ is an eigenvalue with corresponding eigenvector $\boldsymbol{v} \in F^n$ if

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{v} \neq \boldsymbol{0}.$$

Example 5.2 Let $V = \mathbb{R}^2$ and consider the matrix

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$

We calculate that

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A with eigenvalue 2 and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for A with eigenvalue -4.

Finding eigenvalues and eigenvectors

Let $T \colon V \to V$ and let $A = \mathrm{Mat}(T)$ be the matrix of T with respect to some basis $\mathscr{B}$ for V. We seek to find a scalar λ and a non-zero vector $v \in V$ such that $Tv = \lambda v$. This rearranges to

$$(\lambda I - T)v = \mathbf{0}$$

and correspondingly we find

$$(\lambda I - A)\boldsymbol{v} = \mathbf{0}$$

for some $\boldsymbol{v} \neq \mathbf{0}$ in F^n. (The entries of $\boldsymbol{v}$ are precisely the coefficients of v when expressed as a linear combination of the vectors in the basis $\mathscr{B}$.) For such a non-zero vector $\boldsymbol{v}$, the matrix $\lambda I - A$ cannot be invertible. Hence

$$\lambda \text{ is an eigenvalue of } T \qquad \text{if and only if} \qquad \det(\lambda I - A) = 0.$$

We therefore make the following definition.

Definition 5.3 Let $T \colon V \to V$ be a linear transformation and let A be the matrix of T with respect to some basis $\mathscr{B}$ for V. The *characteristic polynomial* of T is

$$\det(xI - A);$$

that is, we expand the determinant of the matrix involving the variable x to obtain a polynomial in x.

It is a fact (delayed until MT3501) that the characteristic polynomial does not depend on the choice of basis $\mathscr{B}$ for V.

We have observed that the eigenvalues of a linear transformation are the roots of its characteristic polynomial. (Note that this is *not* the definition of the eigenvalues: it is the *method* to find the eigenvalues. The definition of eigenvalue — given in Definition 5.1 — makes sense even over infinite-dimensional vector spaces while we cannot calculate a determinant of a matrix in such a setting.) We then find the eigenvectors by solving the equation $Tv = \lambda v$ for each eigenvalue λ that we have determined.

In the examples that follow, we shall consider a matrix A as a linear transformation $A \colon F^n \to F^n$ (as in Example 4.2). Then the matrix of the linear transformation A with respect to the standard bases is A itself and so we find the eigenvalues by solving the equation $\det(xI - A) = 0$.

Example 5.4 *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 7 & 3 \\ -2 & -4 & -2 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is

$$\begin{aligned} \det(xI - A) &= \det \begin{pmatrix} x-1 & 4 & 0 \\ -2 & x-7 & -3 \\ 2 & 4 & x+2 \end{pmatrix} \\ &= (x-1)\big((x-7)(x+2)+12\big) - 4\big(-2(x+2)+6\big) \\ &= (x-1)(x^2-5x-14+12) - 4(2-2x) \\ &= (x-1)(x^2-5x-2) + 8(x-1) \\ &= (x-1)(x^2-5x-2+8) \\ &= (x-1)(x^2-5x+6) \\ &= (x-1)(x-2)(x-3). \end{aligned}$$

Hence the roots of the characteristic polynomial are 1, 2 and 3 and these are the eigenvalues of A.

We now solve the equation $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for each eigenvalue λ in turn seeking a non-zero solution $\boldsymbol{v}$.

Case $\boldsymbol{\lambda = 1}$: Our equation $A\boldsymbol{v} = \lambda\boldsymbol{v} = \boldsymbol{v}$ rearranges to $(A - I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 0 & -4 & 0 \\ 2 & 6 & 3 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we arrive at three equations:

$$-4y = 0, \qquad 2x + 6y + 3z = 0, \qquad -2x - 4y - 3z = 0.$$

The first tells us that $y = 0$ and then the last two both reduce to $2x + 3z = 0$. (This is a general phenomenon of this process. The fact that λ is an eigenvector will always ensure that there is a non-zero solution and hence some redundancy in the equations.) We can take x to be any non-zero value to obtain a non-zero solution of our equation. We shall choose $x = 3$ for then $z = -2x/3 = -2$. Hence

$$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } 1.$$

Case $\boldsymbol{\lambda} = 2$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} -1 & -4 & 0 \\ 2 & 5 & 3 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this equation, let us apply row operations to the matrix (and the vector appearing on the right-hand side, though since it is zero we shall perceive no change!):

$$\left(\begin{array}{ccc|c} -1 & -4 & 0 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & -4 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -1 & -4 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right) \qquad \begin{array}{l} r_2 \mapsto r_2 + 2r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{array}$$

Hence we obtain essentially two equations:

$$-x - 4y = 0, \qquad y - z = 0.$$

(Note that both the second and third rows give rise to scalar multiples of the latter equation.) Given any non-zero choice of y, we now obtain a solution. We shall choose $y = 1$, so that $x = -4y = -4$ and $z = y = 1$. Hence

$$\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } 2.$$

Case $\boldsymbol{\lambda} = 3$: We solve $(A - 3I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} -2 & -4 & 0 \\ 2 & 4 & 3 \\ -2 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we obtain three equations

$$-2x - 4y = 0, \qquad 2x + 4y + 3z = 0, \qquad -2x - 4y - 5z = 0$$

which immediately reduce to

$$x + 2y = 0, \qquad z = 0.$$

Here we shall take $y = 1$, so that $x = -2$ and $z = 0$. Hence

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } 3.$$

Note that in each case, we always have some choice for our eigenvector. This reflects the (easily verified) fact that if v is an eigenvector with eigenvalue λ for a linear transformation T, then any *non-zero* scalar multiple of v is also an eigenvector with the same eigenvalue.

Example 5.5 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: First we find the characteristic polynomial:

$$\begin{aligned} \det(xI - A) &= \det \begin{pmatrix} x-2 & -1 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix} \\ &= (x-2)(x-1)^2. \end{aligned}$$

Hence the eigenvalues (roots of the characteristic polynomial) are

$$\lambda = 2 \text{ and } 1 \text{ (twice)}.$$

We now find the eigenvectors associated to each eigenvalue.

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We deduce that $y = z = 0$, while x may be arbitrary. Hence

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue 2.}$$

Case $\boldsymbol{\lambda = 1}$: We solve $(A - I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $x + y + z = 0$. Therefore two of the variable, say x and y, can be arbitrary and the third is then determined. Taking $x = 1$ and $y = 0$ gives the eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

while taking $x = 0$ and $y = 1$ gives the eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

It is easy to check that these last two vectors are linearly independent (for example, neither is a scalar multiple of the other). Hence, although we have a repeated eigenvalue, we have managed to find as two *linearly independent* eigenvalues with eigenvalue 1.

It is not always the case that we can find as many linearly independent eigenvectors as the eigenvalue occurs as a repeated root of the characteristic polynomial. Investigating this situation is one of the most important topics in linear algebra and it will be considered in greater detail in MT3501.

Change of basis

Recall the matrix A from Example 5.4:

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 7 & 3 \\ -2 & -4 & -2 \end{pmatrix}.$$

In that example, we found three eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

with eigenvalues 1, 2 and 3, respectively. Using the methods from earlier in the course, it is not difficult to verify that these three vectors are linearly independent. Hence $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$. We can therefore consider the matrix of the linear transformation $A\colon \boldsymbol{v} \mapsto A\boldsymbol{v}$ with respect to the basis $\mathscr{B}$. However, the three vectors in $\mathscr{B}$ are *eigenvectors*, so when we express the image of each of them in terms of $\mathscr{B}$ we see that

$$A\boldsymbol{v}_1 = \boldsymbol{v}_1, \quad A\boldsymbol{v}_2 = 2\boldsymbol{v}_2, \quad A\boldsymbol{v}_3 = 3\boldsymbol{v}_3.$$

We write the coefficients down the columns of the matrix:

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

This is a diagonal matrix and this is precisely the point behind finding eigenvectors. If we can find a basis $\mathscr{B}$ for our vector space consisting of eigenvectors for a linear transformation T, then the matrix of T with respect to $\mathscr{B}$ is a diagonal matrix.

But how is this new matrix related to the original one? We shall now address this.

Fix a vector space V over a field F and a linear transformation $T\colon V \to V$. Consider two different bases $\mathscr{A}$ and $\mathscr{B}$ for V. We shall determine how $\mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T)$ and $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ are related. Let $\mathscr{A} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{B} = \{w_1, w_2, \dots, w_n\}$. Suppose that

$$\mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T) = [\alpha_{ij}] \qquad \text{and} \qquad \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = [\beta_{ij}].$$

This means that

$$T(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i \qquad \text{and} \qquad T(w_j) = \sum_{i=1}^{n} \beta_{ij} w_i$$

for $j = 1, 2, \dots, n$.

To determine how the α_{ij} and β_{ij} are related, the important thing to remember that any vector in V can be uniquely expressed as a linear combination of the members of a basis. In particular, we can write

$$w_j = \sum_{k=1}^{n} \lambda_{kj} v_k \tag{5.1}$$

and

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} w_i \tag{5.2}$$

(for some coefficients $\lambda_{kj}, \mu_{i\ell} \in F$), so expressing each basis vector w_j from $\mathscr{B}$ in terms of the basis $\mathscr{A}$ and *vice versa*. From this, we calculate

$$\begin{aligned}
T(w_j) &= T\biggl(\sum_{k=1}^{n} \lambda_{kj} v_k\biggr) \\
&= \sum_{k=1}^{n} \lambda_{kj} T(v_k) \\
&= \sum_{k=1}^{n} \lambda_{kj} \sum_{\ell=1}^{n} \alpha_{\ell k} v_\ell \\
&= \sum_{\ell=1}^{n} \sum_{k=1}^{n} \alpha_{\ell k} \lambda_{kj} \sum_{i=1}^{n} \mu_{i\ell} w_i \\
&= \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj} w_i \\
&= \sum_{i=1}^{n} \biggl(\sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj}\biggr) w_i.
\end{aligned}$$

This must be the *unique* expression for $T(w_j)$ as a linear combination of the vectors in the basis $\mathscr{B}$. Hence

$$\beta_{ij} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj}.$$

This formula is simply that expressing the multiplication of the matrices involved. Specifically, if we write

$$A = \mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T) = [\alpha_{ij}], \qquad B = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = [\beta_{ij}]$$
$$P = [\lambda_{ij}], \qquad Q = [\mu_{ij}],$$

then the above formula says

$$B = QAP.$$

However, it turns out that Q and P are also linked. Substituting (5.1) into (5.2) gives

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} \sum_{k=1}^{n} \lambda_{ki} v_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{ki} \mu_{i\ell} \right) v_k.$$

This must be the unique expression for v_ℓ as a linear combination of the vectors in $\mathscr{A} = \{v_1, v_2, \dots, v_n\}$. Thus

$$(PQ)_{kl} = \sum_{i=1}^{n} \lambda_{ki} \mu_{i\ell} = \delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

(This $\delta_{k\ell}$ is called the *Kronecker delta.*) So

$$PQ = I,$$

the $n \times n$ identity matrix. Similarly, substituting (5.2) into (5.1) yields $QP = I$ by the same argument. Hence

$$Q = P^{-1}.$$

We have proved:

Theorem 5.6 *Let V be a vector space of dimension n over a field F and let $T \colon V \to V$ be a linear transformation. Let $\mathscr{A}$ and $\mathscr{B}$ be bases for V and let A and B be the matrices of T with respect to $\mathscr{A}$ and $\mathscr{B}$, respectively. Then there exists an invertible matrix P such that*

$$B = P^{-1}AP.$$

The coefficient in the (i,j)th entry of P is found by writing each vector w_j in the basis $\mathscr{B}$ in terms of the vectors v_i appearing in the basis $\mathscr{A}$.

We mention briefly that precisely the same sort of argument (though slightly more complicated because two change of bases are involved) establishes what happens if we have a linear transformation $T \colon V \to W$ and we consider different bases for V and for W:

Theorem 5.7 *Let V and W be finite-dimensional vector spaces over a field F and let $T\colon V \to W$ be a linear transformation. Suppose that $\mathscr{B}$ and $\mathscr{B}'$ are bases for V and $\mathscr{C}$ and $\mathscr{C}'$ be bases for W. Then there exist invertible matrices P and Q such that*

$$\mathrm{Mat}_{\mathscr{B}',\mathscr{C}'}(T) = Q^{-1} \cdot \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T) \cdot P.$$

Moreover, the (i,j)th entry of P is the coefficient when the jth vector of $\mathscr{B}$ is written in terms of the basis $\mathscr{B}'$ and the (i,j)th entry of Q is the coefficient when the jth vector of $\mathscr{C}$ is written in terms of the basis $\mathscr{C}'$.

Example 5.8 *Let*

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 7 & 3 \\ -2 & -4 & -2 \end{pmatrix}.$$

Find a 3×3 matrix P such that $P^{-1}AP = D$ is a diagonal matrix.

Solution: This is the matrix we considered in Example 5.4. We found three eigenvectors, namely

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

which have eigenvalues 1, 2 and 3, respectively. Let $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and let $\mathscr{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. If P is the change of basis matrix from $\mathscr{A}$ to $\mathscr{B}$, then Theorem 5.6 says that $P^{-1}AP$ is the matrix of A with respect to $\mathscr{B}$, namely

$$P^{-1}AP = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

To calculate the matrix P, we write each vector $\boldsymbol{v}_i$ in $\mathscr{B}$ in terms of the standard basis:

$$\begin{aligned} \boldsymbol{v}_1 &= \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = 3\mathbf{e}_1 \qquad - 2\mathbf{e}_3 \\ \boldsymbol{v}_2 &= \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = -4\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \boldsymbol{v}_3 &= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2\mathbf{e}_1 - \mathbf{e}_2 \end{aligned}$$

We write these coefficients down the columns of P:

$$P = \begin{pmatrix} 3 & -4 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

Definition 5.9 A linear transformation $T\colon V \to V$ of a finite dimensional vector space V is said to be *diagonalisable* if there is a basis for V with respect to which the matrix of T is represented by a diagonal matrix.

If A is an $n \times n$ matrix with entries from F, then it determines a linear transformation $A\colon F^n \to F^n$. If P is the change of matrix from the standard basis to another basis $\mathscr{B}$, then we have observed the matrix of A with respect to the new basis $\mathscr{B}$ is $P^{-1}AP$. The corresponding definition for matrices is then:

Definition 5.10 Let A be a square matrix over a field F. We say A is *diagonalisable* if there is an invertible matrix P such that $P^{-1}AP$ is diagonal.

A linear transformation T is diagonalisable if there is a basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ such that the matrix of T with respect to $\mathscr{B}$ has the form

$$\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

This means that $T(v_i) = \lambda_i v_i$ for each i. Consequently:

Theorem 5.11 *A linear transformation $T\colon V \to V$ is diagonalisable if and only if there is a basis for V consisting of eigenvectors for T.* □

Example 5.12 *Let*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Solution: In Example 5.5, we determined the eigenvalues and linearly independent eigenvectors for this matrix, namely

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

with eigenvalues 2, 1 and 1 respectively. The entries appearing in these vectors are the coefficients when we write then in terms of the standard basis for $\mathbb{R}^3$. Hence the change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

and $P^{-1}AP$ is the diagonal matrix whose entries are the eigenvalues:

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[Exercise: Check via hand calculation this equation.]

Powers of matrices

The advantage of diagonal matrices is that it is much easier to calculate powers of a diagonal matrix. Suppose that A is a matrix which is diagonalisable, say $P^{-1}AP = D$, where D is diagonal. Rearranging we have

$$A = PDP^{-1}.$$

Calculating successive powers:

$$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$$

and

$$A^3 = PDP^{-1} \cdot PDP^{-1} \cdot PDP^{-1} = PD^3P^{-1},$$

etc. Thus, we can calculate powers of A by calculating powers of D and multiplying by P and P^{-1}. Calcuating powers of D is very easy, for if

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix},$$

then

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n^m \end{pmatrix}.$$

So provided we have found all the eigenvalues λ_i of A and the change of basis matrix P (via calculating the eigenvectors), we can calculate A^m very quickly. (Far more quickly than performing all the matrix multiplications!)

Symmetric matrices

Definition 5.13 An $n \times n$ matrix A is called *symmetric* if $A^{\mathrm{T}} = A$.

Let A be an $n \times n$ symmetric matrix with real entries. It is a fact that such a matrix is diagonalisable. Let $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors for A and let λ_i be the eigenvalue corresponding to the eigenvector $\boldsymbol{v}_i$. Thus

$$A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \qquad \text{for } i = 1, 2, \dots, n. \tag{5.3}$$

Proposition 5.14 *If $\lambda_i \neq \lambda_j$, then $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal; that is,*

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0,$$

where $\cdot$ denotes the usual scalar (or "dot") product for vectors in $\mathbb{R}^n$.

Proof: Recall that

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j$$

where the right-hand side denotes the matrix multiplication of the vector $\boldsymbol{v}_i$ (which is then a $1 \times n$ matrix!) and the transpose of the vector $\boldsymbol{v}_j$ (an $n \times 1$ matrix). Taking the transpose of Equation (5.3) gives

$$\boldsymbol{v}_i^{\mathrm{T}} A = \boldsymbol{v}_i^{\mathrm{T}} A^T = \lambda_i \boldsymbol{v}_i^{\mathrm{T}}.$$

Hence

$$\boldsymbol{v}_i^{\mathrm{T}} A \boldsymbol{v}_j = \lambda_i \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j.$$

On the other hand, using the fact that $\boldsymbol{v}_j$ is also an eigenvector, we deduce

$$\boldsymbol{v}_i^{\mathrm{T}} A \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \lambda_j \boldsymbol{v}_j = \lambda_j \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j.$$

Subtracting these last two equations gives

$$(\lambda_i - \lambda_j) \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = 0.$$

Since $\lambda_i \neq \lambda_j$, we can divide by $\lambda_i - \lambda_j$ and conclude

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = 0,$$

as claimed. □

This establishes that eigenvectors for a symmetric matrix corresponding to distinct eigenvalues are orthogonal. In fact, it can be established that if A is a symmetric $n \times n$ matrix, then there is a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors for A.

Let $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors for A. Let

$$k_i = |\boldsymbol{v}_i| = \sqrt{\boldsymbol{v}_i \cdot \boldsymbol{v}_i}$$

and replace each $\boldsymbol{v}_i$ by $\frac{1}{k_i}\boldsymbol{v}_i$. This has the consequence that each vector $\boldsymbol{v}_i$ now has unit length. Thus we may assume $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an *orthonormal set*:

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let P be the change of basis matrix from the standard basis of $\mathbb{R}^n$ to $\mathscr{B}$; that is, we write the entries of each vector $\boldsymbol{v}_i$ down the columns of P. Thus

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n \end{pmatrix}$$

(the ith column of P is the vector $\boldsymbol{v}_i$). Consider the product $P^{\mathrm{T}}P$:

$$\begin{aligned} P^{\mathrm{T}}P &= \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_2 & \ldots & \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_n \\ \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_2 & \ldots & \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \end{aligned}$$

Hence $P^{\mathrm{T}}P = I$, the identity matrix. Rearranging, we deduce that $P^{-1} = P^{\mathrm{T}}$.

Therefore, applying change of basis to the matrix A we conclude

$$P^{\mathrm{T}}AP = D$$

where D is the diagonal matrix containing the eigenvalues of A.

Definition 5.15 A matrix P whose inverse is equal to its transpose is called *orthogonal*.

We have observed that if A is a real symmetric matrix, then the change of basis matrix P can be taken to be orthogonal:

Theorem 5.16 *If A is a real symmetric matrix, then there exists an orthogonal matrix P such that $P^{\mathrm{T}}AP = D$ is diagonal.* □

Example 5.17 *Find an orthogonal matrix which diagonalises*

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is

$$\begin{aligned} \det \begin{pmatrix} x-3 & -1 \\ -1 & x-3 \end{pmatrix} &= (x-3)^2 - 1 \\ &= x^2 - 6x + 8 \\ &= (x-2)(x-4). \end{aligned}$$

Hence the eigenvalues of A are 2 and 4. We must now find orthonormal eigenvectors.

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $x + y = 0$. Hence

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 2. We now normalise:

$$\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1^2 + (-1)^2 = 2.$$

Hence

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 2 and *unit length.*

Case $\boldsymbol{\lambda = 4}$: We solve $(A - 4I)\boldsymbol{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $x - y = 0$. Hence

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 4. Its length is $\sqrt{1^2+1^2} = \sqrt{2}$, so

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

is an eigenvector for A with eigenvalue 4 and unit length.

Hence

$$\left\{\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right\}$$

is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A. The change of basis matrix is

$$P = \begin{pmatrix}1/\sqrt{2} & 1\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}.$$

[Let us verify that this matrix P does indeed solve the problem:

$$P^{\mathrm{T}}P = \frac{1}{2}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = I.$$

Then

$$\begin{aligned}P^{\mathrm{T}}AP &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\1 & 3\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}2 & -2\\4 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}4 & 0\\0 & 8\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 4\end{pmatrix},\end{aligned}$$

which is indeed the diagonal matrix containing the eigenvalues of A.]

Hermitian matrices

We now describe an analogous situation to that just presented, but now for matrices with entries being complex numbers.

Definition 5.18 Let A be a matrix whose entries are complex numbers. Write $A^\dagger$ for the matrix obtained by taking the complex conjugate of each entry of A and then the transpose of the resulting matrix. That is,

$$A^\dagger = (\bar{A})^{\mathrm{T}}.$$

We previously considered real symmetric matrices and observed that they could be diagonalised using orthogonal matrices. The corresponding types of matrices to be considered here are:

Definition 5.19 (i) A *Hermitian matrix* is a matrix A with complex numbers as entries such that $A^\dagger = A$.

(ii) A *unitary matrix* is a matrix U whose inverse is $U^\dagger$.

Thus, a unitary matrix is a square matrix U satisfying

$$UU^\dagger = U^\dagger U = I.$$

In the same way that a real symmetric matrix can be diagonalised by an orthogonal matrix, here a Hermitian matrix can be diagonalised by a unitary matrix:

Theorem 5.20 *If A is a Hermitian matrix, then there exists a unitary matrix U such that $U^\dagger AU = D$ is diagonal.*

Example 5.21 *Find a unitary matrix that diagonalises the Hermitian matrix*

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Note that

$$\bar{A} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

so $A^\dagger = (\bar{A})^{\mathrm{T}} = A$. Thus the previous theorem does indeed apply here. The method of solution is the same as for real symmetric matrices: we find eigenvectors that are of unit length.

Solution: The characteristic polynomial is

$$\begin{aligned} \det \begin{pmatrix} x-1 & -i \\ i & x-1 \end{pmatrix} &= (x-1)^2 + i^2 \\ &= x^2 - 2x + 1 - 1 \\ &= x^2 - 2x = x(x-2). \end{aligned}$$

Hence the eigenvalues of A are 0 and 2.

Case $\boldsymbol{\lambda = 0}$: We solve $A\boldsymbol{v} = \boldsymbol{0}$:

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $z + iw = 0$. Hence

$$\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue 0. Its magnitude is

$$\sqrt{|-i|^2 + |1|^2} = \sqrt{2},$$

so the unit eigenvector is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $-z + iw = 0$. Hence

$$\begin{pmatrix} i \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue 2 and the unit eigenvector is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

The required unitary matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

[Let us now verify the claim. Note

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix},$$

so

$$U^\dagger U = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

(using the fact that $i^2 = -1$). Then

$$\begin{aligned} U^\dagger A U &= \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -2i & 2 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \end{aligned}$$

which (as required) is a diagonal matrix whose diagonal entries are the eigenvalues of A.]

We final fact, which we state without proof and which was a feature in the previous example, is:

Proposition 5.22 *If A is a Hermitian matrix, then all its eigenvalues are real numbers.*