MRQ 2006

School of Mathematics and Statistics

MT1003 Pure Mathematics

Problem Sheet I: Divisibility of Integers

1. For each of the following statements decide whether it is true or false:

(i) $5 \mid 15$; (ii) $-7 \mid 15$; (iii) $8 \mid 256$; (iv) $4 \nmid -12$;
(v) $7 \mid 0$.

2. Let a, b, c, x and y be any integers. Prove the following statements:

(a) If $a \mid b$ and $b \mid c$, then $a \mid c$.

(b) If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$.

3. For each of the following statements about arbitrary integers a, b and c, either provide a proof or a counterexample.

(a) If $a \mid b$, then $a \mid bc$.

(b) If $a \mid (b + c)$, then $a \mid b$ and $a \mid c$.

(c) If $a \mid bc$, then $a \mid b$ or $a \mid c$.

(d) If $a \mid b$ and $a \mid c$, then $a^2 \mid bc$.

4. Prove that for every integer n the number $n(n + 1)(2n + 1)/6$ is an integer.

5. Let n be a positive integer. Prove that for any x we have

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Hence prove that the number $2^{3n} - 1$ is divisible by 7.

6. Write the number 113 in (i) base 2; (ii) base 5.

7. Write the number 237 in (i) base 3; (ii) base 16.

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Problem Sheet II: Greatest Common Divisors and the Euclidean Algorithm; Primes and Factorisation

1. Find $\gcd(143, 227)$, $\gcd(306, 657)$, and $\gcd(272, 1479)$.

2. Use the Euclidean Algorithm to find integers x and y such that

(a) $\gcd(56, 72) = 56x + 72y$;

(b) $\gcd(24, 138) = 24x + 138y$;

(c) $\gcd(119, 273) = 119x + 273y$.

3. Prove that if $d = \gcd(a, b)$, then $\gcd(a/d, b/d) = 1$.

4. Factorise the following numbers into products of primes: (i) 3456; (ii) 10140; (iii) 36000; (iv) 97461.

5. Find all prime numbers dividing $50!$.

6. Prove that for every $k \geqslant 2$ there exists a number with precisely k divisors.

7. Prove that there are infinitely many primes of the form $6k + 5$.

8. Find all primes of the form $n^3 - 1$.

9. Find all prime numbers p such that $p + 2$ and $p + 4$ are also prime.

[Hint: First find one prime p_1 which satisfies this property. For an arbitrary prime satisfying this property, consider its remainder modulo p_1.]

10. Find all prime numbers p such that $p^2 + 2$ is also prime.

11. If p is a prime and $p \mid a^n$, prove that $p^n \mid a^n$.

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Problem Sheet III: Linear Diophantine Equations; Congruences

1. For each of the following linear Diophantine equations determine if it has solutions, and if so determine all the solutions:

(a) $56x + 72y = 40$;

(b) $144x + 60y = 54$;

(c) $172x + 20y = 1000$;

(d) $221x + 91y = 117$;

(e) $84x - 436y = 156$.

2. Determine all positive solutions of the following linear Diophantine equations:

(a) $3x + 5y = 61$;

(b) $4x + 7y = 53$;

(c) $20x + 13y = 200$.

3. For each of the following linear Diophantine equations determine the number of solutions in positive integers:

(a) $30x + 17y = 300$;

(b) $54x + 21y = 906$;

(c) $4x + 7y = 1179$.

4. Prove that if a and b are positive integers with $\gcd(a, b) = 1$ and if c is an arbitrary integer then the equation $ax - by = c$ has infinitely many solutions in positive integers.

5. When Mr. Smith cashed a cheque at his bank, the teller mistook the number of pence for the number of pounds, and *vice versa*. Unaware of this, Mr. Smith spent 68p and then noticed that he had twice the amount of the original cheque. Determine the value for which the cheque was originally written.

6. How many positive numbers *cannot* be expressed as $5x + 7y$ with x and y positive?

7. Prove the following facts:

(a) If $a \equiv b \pmod{n}$ and $m \mid n$, then $a \equiv b \pmod{m}$.

(b) If $a \equiv b \pmod{n}$ and $c > 0$, then $ac \equiv bc \pmod{cn}$.

8. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

9. Prove that $3^{6n} - 2^{6n}$ is divisible by 35 for every positive integer n.

10. Prove that every number is congruent to its two-digit ending modulo 100. Decompose $11^{10} - 1$ using $x^n - 1 = (x-1)(x^{n-1} + \cdots + x + 1)$, and show that it is divisible by 100. What are the last two digits of 11^{2000}?

11. Prove that $2^{44} - 1$ is divisible by 89.

12. Prove that $2^{48} - 1$ is divisible by 97.

13. Find the last digit of the numbers (i) 9^{9^9} and (ii) 2^{3^4}.
[Note: 2^{3^4} means $2^{(3^4)}$.]

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Problem Sheet IV: Functions; Equivalence Relations

1. Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ which is:

(a) one-one and onto;

(b) one-one but not onto;

(c) not one-one but onto;

(d) neither one-one nor onto.

2. For each of the following relations R on the set X determine whether it is (i) reflexive, (ii) symmetric, (iii) anti-symmetric, and (iv) transitive.

(a) $X = \{1, 2, \dots, 8\}$; xRy if and only if $x = 2^n y$ for some non-negative integer n.

(b) $X = \{1, 2, \dots, 8\}$; xRy if and only if $x = 2^n y$ for some integer n.

(c) X is the set of all subsets of $\{1, 2, 3\}$; R is the relation '…is a subset of…'.

(d) $X = \mathbb{Z}$; xRy if and only if 3 divides $x + 2y$.

3. Let R be the relation on the set $\mathbb{Q} \setminus \{0\}$ of non-zero rational numbers defined by

$$xRy \quad \text{if and only if} \quad x/y \in \mathbb{Z}.$$

Is this relation (i) reflexive, (ii) symmetric, (iii) anti-symmetric, and (iv) transitive?

4. Give an example of a set and a relation defined upon it that is reflexive and transitive, but is neither symmetric nor anti-symmetric.

5. Define a relation R on the set $\mathbb{R}$ of real numbers by

$$xRy \quad \text{if and only if} \quad x^2 = y^2.$$

Prove that R is an equivalence relation and describe its equivalence classes.

6. Define relations R and S on the set $\mathbb{Z}$ of integers by

$$\begin{aligned} xRy \quad &\text{if and only if} \quad 2 \mid (x+y), \\ xSy \quad &\text{if and only if} \quad 3 \mid (x+y). \end{aligned}$$

Prove that R is an equivalence relation but that S is not an equivalence relation.

7. Define a relation R on the set $\mathbb{Z} \times \mathbb{Z}$ of all ordered pairs of integers by

$$(x,y)R(z,t) \quad \text{if and only if} \quad x+t = y+z.$$

Prove that R is an equivalence relation and describe its equivalence classes.

8. Define a relation R on $\mathbb{R}^2$ by

$$(x_1,y_1)R(x_2,y_2) \quad \text{if and only if} \quad x_1^2+y_1^2 = x_2^2+y_2^2.$$

Prove that R is an equivalence relation. Describe geometrically the equivalence classes.

9. Let $X = \mathbb{Z} \times \mathbb{N}$ be the set of ordered pairs (a,b) where a is any integer and b is a positive integer. Define a relation R on X by

$$(a,b)R(c,d) \quad \text{if and only if} \quad ad = bc.$$

Prove that R is an equivalence relation on X. Describe the collection of equivalence classes of R.

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Problem Sheet V: Pythagorean triples; higher degree Diophantine equations

1. Find all primitive Pythagorean triples (x, y, z) in which (i) $x = 40$; (ii) $x = 60$; (iii) $y = 45$.

2. Find three different Pythagorean triples (x, y, z) (not necessarily primitive) with $x = 16$ and both y and z positive.

3. Prove that if $n \not\equiv 2 \pmod 4$ then there exists a primitive Pythagorean triple (x, y, z) in which x or y equals n.

4. Prove that if (x, y, z) is a Pythagorean triple then $12 \mid xy$ and $60 \mid xyz$. [Hint: It is sufficient to consider the primitive Pythagorean triples.]

5. Find all right-angled triangles with sides of integral length having area 60.

6. Find all right-angled triangles with sides of integral length, whose area is equal to the perimeter.

7. Prove that the equation $x^2 + y^2 = z^3$ has infinitely many solutions in positive integers. [Hint: Find one solution and then multiply.]

8. Prove that the equation

$$x^2 + y^2 = 2xy$$

has infinitely many solutions in integers.

Consider the equation

$$x^2 + y^2 + z^2 = 2xyz.$$

Assume that (x_0, y_0, z_0) is a solution to this equation in integers. Working modulo 4, prove that all x_0, y_0, z_0 must be even. Writing $x_0 = 2x_1,\;\; y_0 = 2y_1,\;\; z_0 = 2z_1$, show that

$$x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1.$$

Conclude that x_1, y_1 and z_1 must be even as well, and that x_0, y_0 and z_0 must be divisible by 4.

Continuing in this way, conclude that $x_0 = y_0 = z_0 = 0$ is the only solution of the original equation.

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Problem Sheet VI: Graphs

1. Seven villages a, b, c, d, e, f and g are connected by a one-way system of roads as follows:

- B22 goes from a to c passing through b;
- B33 goes from c to d and then passes through b as it continues to f;
- B44 goes from d through e to a;
- B55 goes from f to b passing through g; and
- B66 goes from g to d.

(a) Using vertices for villages and directed edges for segments of roads between villages, draw a directed graph that models this situation.

(b) List the simple paths from g to a (that is, paths from g to a that do not pass through any vertex more than once).

(c) What is the smallest number of road segments that would have to be closed down so that travel from b to d is disrupted?

(d) Is it possible to leave village c and return there, visiting each of the other villages only once?

(e) What is the answer to (d) if we are not required to return to c?

(f) Is it possible to start at some village and drive over all of these roads exactly once? (You are entitled to visit a village more than once and you need not return to the village from which you started.)

2. (a) How many non-isomorphic (loop-free) digraphs are there with 3 vertices?

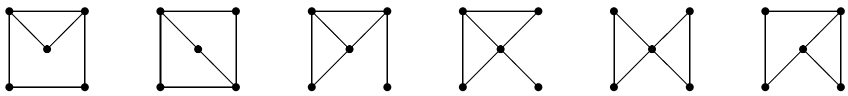
(b) How many non-isomorphic (simple) graphs with 3 vertices?

(c) How many non-isomorphic (simple) graphs with 4 vertices?

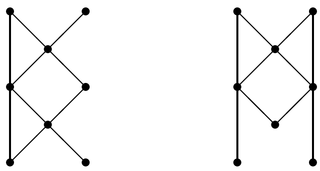
3. A ferryman (F) has to transport a dog (D), a sheep (S) and a bag of cabbage (C) across a river. His boat will only carry one of these items at one time. Furthermore he cannot leave the dog alone with the sheep nor the sheep with the cabbage. Construct a graph whose vertices are the allowable combinations on the bank. Join two vertices if a single trip by the ferryman changes one combination to the other. Hence find the total number of different solutions to the problem.

[Hint: The initial vertex might be labelled FDSC|∅ indicating all four on one side of the river. The final vertex would then be labelled ∅|FDSC, while one permitted vertex would be labelled D|FSC. There should be no vertex labelled FD|SC, since this represents the sheep and cabbage together unsupervised on one side of the river.]

4. For each of the following graphs find the degrees of the vertices. Deduce that, although they have the same number of vertices and edges, only one pair of them is isomorphic.

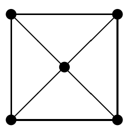


5. Write down an isomorphism between the following two graphs.

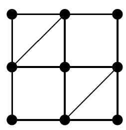


6. Which of the following graphs are (a) Eulerian, (b) Hamiltonian? Justify your assertions.

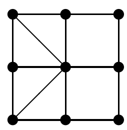
(i)



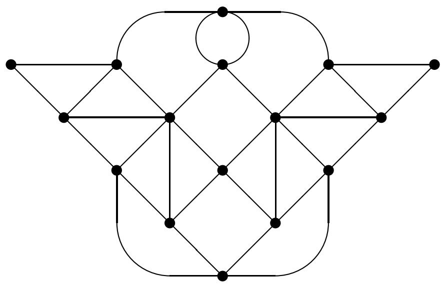
(ii)



(iii)



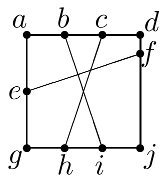
7. Does the following graph have an Eulerian circuit? Does it have a Hamiltonian circuit? Find such circuits if they exist.

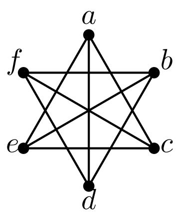


8. By considering a graph with 64 vertices show that it is not possible for a knight to move around an 8×8 chess board making all the possible "knight's moves" between pairs of squares exactly once.

[Note: This is not the same question as asking whether we can visit each square exactly once using "knight's moves".]

9. Determine whether or not the following graphs are planar. If planar, redraw the graph with no edges intersecting except at a vertex. If non-planar, find a subgraph homeomorphic to either K_5 or $K_{3,3}$.

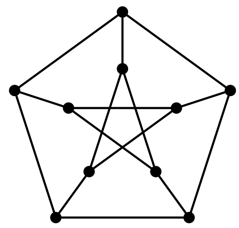
(i) 

(ii) 

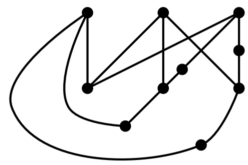
10. Prove that every connected planar simple graph contains at least one vertex of degree 5 or less.

[Hint: It may help to sum the number of edges meeting at each vertex.]

11. The following graph is known as the Petersen graph:



By a suitable labelling of vertices, show that the Petersen graph contains the following graph as a subgraph:



Hence, or otherwise, determine whether or not the Petersen graph is planar.

12. Sketch the (unique) trees corresponding to a saturated hydrocarbon with 2 carbon atoms and with 3 carbon atoms.

Give two non-isomorphic trees corresponding to a saturated hydrocarbon with 4 carbon atoms.

13. Determine the non-isomorphic trees corresponding to a saturated hydrocarbon with 5 carbon atoms.

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Problem Sheet VII: Permutations; Groups

1. Let σ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 2 & 7 & 8 & 1 & 6 \end{pmatrix}$$

and let τ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 5 & 1 & 8 & 6 & 7 & 3 \end{pmatrix}.$$

(a) Calculate $\sigma\tau,\ \tau\sigma,\ \sigma^2,\ \tau^2,\ (\sigma\tau)^2,\ (\tau\sigma)^2,\ \tau^{-1}\sigma\tau$, and $\sigma^{-1}\tau\sigma$.

(b) Find the smallest positive integers m and n such that σ^m and τ^n are the identity permutation.

2. Express the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 1 & 7 & 3 & 8 & 5 \end{pmatrix} \in S_8$$

as a product of disjoint cycles. Find an element of order 2 in S_8 which commutes with σ.

Express σ as a product of transpositions.

3. Consider a square and label its vertices with the numbers 1, 2, 3 and 4. Consider the symmetries of this square. Each symmetry produces a permutation of $\{1, 2, 3, 4\}$ via this labelling. However, there are 24 permutations on $\{1, 2, 3, 4\}$ but only 8 symmetries of the square.

Let σ be a clockwise rotation through $90°$. Write down σ as a permutation of $\{1, 2, 3, 4\}$. Compute σ^2. What symmetry of the square is σ^2?

4. Which of the following sets are groups under the given binary operation?

(a) The even integers under addition.

(b) The rationals under subtraction.

(c) The rationals under multiplication.

(d) The positive rationals under multiplication.

(e) $\{1, 5, 7, 11\}$ under multiplication modulo 12.

(f) $\{\, a + b\sqrt{2} \mid a, b \in \mathbb{Q},\ \ a, b \text{ not both zero} \,\}$ under real multiplication.

5. Show that the set of matrices

$$\left\{ \begin{pmatrix} a + 2b & 3b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R},\ \ a, b \text{ not both zero} \right\}$$

forms a group under multiplication. Determine whether the group is abelian or non-abelian.

6. The following is an incomplete multiplication table of a group G of order 6.

	l	m	n	p	q	r
l						r
m			q			
n					r	
p						
q		l				
r						

Find the identity element of G and show that m, n and r are powers of q. Deduce that every element of the group G is a power of q and so complete the multiplication table.

7. Which of the following groups are abelian? Which are cyclic? Which are finite?

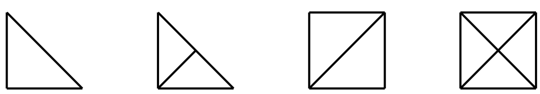
(a) $G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$ under multiplication.

(b) The even integers under addition.

(c) $\mathbb{R}$ under addition.

(d) $\{1, 3, 5, 7\}$ under multiplication modulo 8.

8. Describe the symmetry groups of the following figures:



9. Which of the following are subgroups of the indicated groups?

(a) Is $\mathbb{Z}$ a subgroup of $(\mathbb{Q}, +)$? Is $\mathbb{Z}^+$ (the set of positive integers together with 0) a subgroup of $(\mathbb{Q}, +)$?

(b) Is $\mathbb{Z} \setminus \{0\}$ a subgroup of $(\mathbb{Q}^*, \cdot)$, the multiplicative group of the non-zero rationals?